TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 217, 1976

ON THE TORUS THEOREM AND ITS APPLICATIONS

BY

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Dedicated to Professor R. H. Fox

ABSTRACT. In this paper, we prove the torus theorem and that manifolds in a certain class of 3-manifolds with toral boundary are determined by their fundamental groups alone. Both of these results were reported by F. Waldhausen. We also give an extension of Waldhausen's generalization of the loop theorem.

I. Introduction. The primary purpose of this paper is to prove the "Torus Theorem" reported by F. Waldhausen in [17]. This theorem relates the existence of essential maps of tori into a 3-manifold M and essential embeddings of tori or annuli into M. The reader should note that the manifold M is required to be bounded and irreducible. Originally the author thought that the torus theorem would also hold if one assumed that M was "sufficiently large" but not necessarily bounded. Unfortunately this is not true as is shown by an example communicated to the author by W. Jaco. Jaco obtained his example by surgery on the space of a torus knot of genus greater than two.

The results of this paper also follow from theorems proved independently by Johanssen and by Jaco and Shalen which classify the boundary preserving maps of a torus or annulus into a sufficiently large 3-manifold, up to boundary preserving homotopy.

We obtain as corollaries to the "torus theorem" partial answers to questions of Neuwirth (question T in [8]) and Papakyriakopoulos (question 3 in [9]). The "torus theorem" also relates to a question of R. H. Fox (problem 4 in [7]). In the general case a negative answer to Fox's question has been given in [4] and [13].

We also obtain a result that shows that a certain class of 3-manifolds with toral boundaries are determined by their fundamental groups alone.

In order to establish the "torus theorem", we first prove supporting theorems that have some interest in their own right.

Firstly we give an extension of the "annulus theorem" reported by

Received by the editors March 14, 1974 and, in revised form, April 18, 1975.

AMS (MOS) subject classifications (1970). Primary 55A35, 55A05; Secondary 57C35, 55A25.

Key words and phrases. Torus, annulus, essential map, essential embedding.

(1) The author is partially supported by NSF Grant GP 15357.

Waldhausen in [17] and proved in [6]. We also give an extension of Waldhausen's "generalized loop theorem" [14].

The author acknowledges his indebtedness to F. Waldhausen who announced results in [17] that suggested the direction of this paper.

II. **Notation.** Throughout this paper all spaces are simplicial complexes and all maps are piecewise linear. We use regular neighborhood to mean second regular neighborhood. The closure of a set $X \subset Y$ is denoted by cl(X). All manifolds are compact and orientable unless they arise in a proof or we admit they are nonorientable. We denote the boundary of a manifold M by ∂M and $M - \partial M$ by int(M). A manifold N is *properly embedded* in a manifold M if $N \cap \partial M = \partial N$. If X and Y are connected spaces, the natural map from $\pi_1(X)$ into $\pi_1(Y)$ induced by inclusion is denoted by $\pi_1(X) \longrightarrow \pi_1(Y)$. A two-sided connected surface F properly embedded in a 3-manifold M is *incompressible* (in M) if $\pi_1(F) \longrightarrow \pi_1(M)$ is monic. It is a well-known corollary to the loop theorem [12] that F is incompressible in M if and only if, for each disk $\mathcal{D}$ embedded in M such that $\mathcal{D} \cap F = \partial\mathcal{D}$, $\partial\mathcal{D}$ is nullhomotopic on F. A surface embedded in M is *incompressible* in M if each of its components is incompressible.

Throughout this paper A will denote an annulus and c_1 and c_2 the components of ∂A. An arc properly embedded in A whose complement is simply connected is a *spanning arc* of A. We denote one such arc on A by α. Let F be a surface embedded in the manifold M. A map $f\colon (A, \partial A) \longrightarrow (M, F)$ is *F-essential* if

(1) $f_*\colon \pi_1(A) \longrightarrow \pi_1(M)$ is monic.

(2) The arc $f(\alpha)$ is not homotopic rel its endpoints to an arc on F.

If $F = \partial M$, we say that f is *essential.* The reader should observe that the definition of F-essential is independent of the choice of α.

Throughout this paper T will denote a 2-dimensional torus and $\mathcal{D}$ will denote a 2-dimensional disk. Let M be a 3-manifold and $f\colon T \longrightarrow M$ a map. Then f is *essential* if

(1) $f_*\colon \pi_1(T) \longrightarrow \pi_1(M)$ is monic.

(2) There is an element $\sigma \in \pi_1(T)$ such that $f_*(\sigma)$ has a representative loop not freely homotopic to a loop in ∂M.

We define an essential map of a Klein bottle in the same way. We shall say that f is *S-essential* if f is essential and for each $\sigma \neq 1 \in \pi_1(T)$, $f_*(\sigma)$ has a representative not freely homotopic to a loop in ∂M. If $f\colon T \longrightarrow M$ is an essential map that is not S-essential, we say that f is *W-essential.*

Let F_1 and F_2 be disjoint surfaces properly embedded in a 3-manifold M. Then F_1 is *parallel* to F_2 if there is an embedding $g\colon F_1 \times [0, 1] \longrightarrow M$ such that

(1) $g(F_1 \times \{0\}) = F_1$.

(2) $g(F_1 \times \{1\}) = F_2$.

(3) $g(\partial F_1 \times [0, 1]) \subset \partial M$.

Let $f: X \longrightarrow Y$ be a map. Then we denote $\mathrm{cl}\{x \in X: f^{-1}f(x) \neq \{x\}\}$ by $S(f)$. We use the terms *hierarchy* and *boundary incompressible* as they are defined in [15] and [16]. We shall denote the Euler characteristic of a surface F by $\chi(F)$. Let M be a 3-manifold with incompressible boundary. Let F be a two-sided, connected surface properly embedded in M. We shall say that F is *good* (*with respect to M*) if

(1) F is incompressible.

(2) F is boundary incompressible.

(3) F does not separate M.

(4) A two-sided surface F_1 properly embedded in M and satisfying properties (1), (2) and (3) above cannot be found so that $\chi(F_1) > \chi(F)$.

(5) A two-sided surface F_1 properly embedded in M and satisfying properties (1), (2), (3) and (4) above cannot be found so that $\mathrm{genus}(F_1) < \mathrm{genus}(F)$.

Let $f: T \longrightarrow M_1$ be an essential map. Let M_j, $F_j \subset M_j$, $U(F_j)$ for $j = 1, \ldots, n$ be a hierarchy for M_1. We suppose that

(1) F_j is a disk if ∂M_j is compressible for $j = 1, \ldots, n$.

(2) F_j is an essential annulus if ∂M_j is incompressible and M_j admits an essential embedding of an annulus for $j = 1, \ldots, n$.

(3) F_j is a good surface otherwise.

We assume that if there is a map f_1 homotopic to f and a surface F_1' satisfying (1)–(3) above such that $f_1(T) \cap F_1'$ is empty, F_1 has been chosen to be such a surface. More generally if there is a map f_1 homotopic to f such that $f_1(T) \subset M_j$ and surface F_j' in M_j satisfying (1)–(3) above such that $f_1(T) \cap F_j'$ is empty, F_j is such a surface for $j = 1, \ldots, n$.

We shall say that a hierarchy having the properties above is *special with respect to f.*

Let M_1 be an orientable 3-manifold and F a two-sided connected surface properly embedded in M_1. Then the *manifold M_2 obtained by splitting M_1 along F* has by definition the property: ∂M_2 contains surfaces F_1 and F_2 which are copies of F, and identifying F_1 and F_2 gives a projection $P: (M_2, F_1 \cup F_2) \longrightarrow (M_1, F)$.

We say that a map $f: A \longrightarrow M$ is *transverse with respect to a surface F properly embedded in M* if for each $x \in f^{-1}(F)$

(1) $f^{-1}f(x)$ contains at most two points.

(2) Let R_1 be a regular neighborhood of F.

Let R_2 be a regular neighborhood of $x \in A$. Then $R_2 - f^{-1}(F)$ has two components and f carries these components to distinct components of $R_1 - F$.

Let F be a surface properly embedded in a 3-manifold M. Let F_1 and F_2

be disjoint surfaces embedded in M so that

$$F_j \cap (F \cup \partial M) = \partial F_j \quad \text{for } j = 1, 2.$$

We say that F_1 and F_2 are *parallel* rel F if the surfaces in $P^{-1}(F_1 \cup F_2)$ are parallel in the manifold obtained by splitting M along F.

Let $\mathcal{K}$ be a knot in S^3. Let M be the complement of the interior of a regular neighborhood of $\mathcal{K}$ in S^3. Then we shall say that M is the *knot space* of $\mathcal{K}$.

III. F-essential annuli. In this section we relate the existence of F-essential maps and F-essential embeddings.

LEMMA 3.1. *Let F be an incompressible, boundary incompressible surface properly embedded in the 3-manifold M. Let $f: \mathcal{D} \longrightarrow M$ be a map such that $f^{-1}(F)$ is an arc β in $\partial\mathcal{D}$ and $f^{-1}(\partial M) = \partial\mathcal{D} - \text{int}(\beta)$. Then there exists a map $f': \mathcal{D} \longrightarrow M$ such that $f' \mid \beta = f \mid \beta$, $f'(\mathcal{D}) \subset F$ and $f'(\partial\mathcal{D} - \beta) \subset \partial M$.*

PROOF. This lemma is a weakened form of Lemma (1.7) in [16] and its proof follows immediately from the proof of (1) in that lemma.

PROPOSITION 3.2. *Let M be a 3-manifold and F an incompressible surface in M. Let $\pi_2(M) = 0$ and $f: (A, \partial A) \longrightarrow (M, F)$ a map such that $f_*: \pi_1(A) \longrightarrow \pi_1(M)$ is monic. Then f is F-essential if and only if it is not homotopic* rel ∂A *to a map into F.*

PROOF. Clearly if f is homotopic to a map into F, f is not F-essential. Suppose f is not F-essential. Then there is a homotopy of $f \mid \alpha$ rel $\partial\alpha$ to a map into F. Thus we may assume that $f(\alpha)$ lies on F. If we split A along α, we obtain a disk $\mathcal{D}$ and f induces a map $f_1: (\mathcal{D}, \partial\mathcal{D}) \longrightarrow (M, F)$. It can be seen that if f_1 is homotopic rel $\partial\mathcal{D}$ to a map into F, f is homotopic rel ∂A to a map into F. Since F is incompressible, $f_1(\partial\mathcal{D})$ is inessential on F. Thus there is a map $f_2: \mathcal{D} \longrightarrow F$ such that $f_2 \mid \partial\mathcal{D} = f_1 \mid \partial\mathcal{D}$. Since $\pi_2(M) = 0$, f_2 is homotopic to f_1 rel $\partial\mathcal{D}$. This completes the proof of Proposition 3.2.

THEOREM 1. *Let M be a 3-manifold and $F \subset \partial M$ an incompressible surface in M. Let $f: (A, \partial A) \longrightarrow (M, F)$ be an F-essential map such that $f \mid \partial A$ is an embedding. Then there is an F-essential embedding $g: (A, \partial A) \longrightarrow (M, F)$ such that $f(\partial A) = g(\partial A)$.*

PROOF. Theorem 1 is an immediate consequence of Theorem 1′ in [6].

THEOREM 2. *Let M be a 3-manifold and F a surface embedded in ∂M such that $\pi_1(F) \longrightarrow \pi_1(M)$ is monic. Let $f: (A, \partial A) \longrightarrow (M, F)$ be an F-essential map. Then there is an F-essential embedding $g: (A, \partial A) \longrightarrow (M, F)$. Furthermore if $f(c_1) \cap f(c_2)$ is empty we may assume that $g(c_j)$ lies in any prespecified neighborhood of $f(c_j)$ for $j = 1, 2$.*

PROOF. One uses the proof of Theorem 2 or Theorem 3 in [6] to obtain an F-essential map $f_1: (A, \partial A) \longrightarrow (M, F)$ such that $f_1 | \partial A$ is a homeomorphism. Theorem 2 can now seem to be a consequence of Theorem 1.

IV. A generalization of the loop theorem. In this section we concern ourselves with replacing a map $f: (A, \partial A) \longrightarrow (M, F)$ by an embedding $g: (A, \partial A) \longrightarrow (M, F)$ so that the class of the loop $g(c_1)$ preserves a certain property of the loop $f(c_1)$. In particular if F is an incompressible surface in ∂M and $f(c_1)$ is not freely homotopic to a loop in ∂F, we will show that we may suppose that g may be chosen so that $g(c_1)$ is not freely homotopic to that loop in ∂M as long as $f(c_1) \cap f(c_2)$ is empty. In case the component F_1 of F on which $f(c_1)$ lies is planar and $\chi(F_1) = -1$, Theorem 3 shows that g may be chosen so that $g(c_1)$ is freely homotopic to any prespecified component of ∂F_1.

In Theorem 3 below we are interested only in the conjugacy class of $[f(c_1)], \tau_1, \ldots, \tau_n$ so we need not concern ourselves with a basepoint for $\pi_1(F)$.

THEOREM 3. *Let F be an incompressible bounded surface embedded in ∂M. Let $\tau_1, \ldots, \tau_n$ be elements in $\pi_1(F)$ with the following properties:*

(1) $\tau_i \neq \sigma^k$ *for all* $\sigma \in \pi_1(F)$, $k \neq \pm 1$, *and* $i = 1, \ldots, n$.

(2) $\sigma_1 \tau_i^u \sigma_1^{-1} \sigma_2 \tau_j^v \sigma_2^{-1} \neq \sigma_3 \tau_k^w \sigma_3^{-1}$ *for all integers i, j, k, u, v, and w and all elements* $\sigma_1, \sigma_2, \sigma_3 \in \pi_1(F)$ *such that* $1 \leqslant i, j, k \leqslant n$, $\sigma_1 \neq \sigma_2$, *and u, v, and w nonzero.*

(3) $\tau_i^u \tau_j^v \neq \sigma \tau_k^w \sigma^{-1}$ *where* $\sigma \in \pi_1(F)$ *unless* $i = j = k$, $\sigma = 1$, *and* $u + v = w$ *where u, v, and w are not equal to zero.*

Let $f: (A, \partial A) \longrightarrow (M, \partial M)$ *be a map such that*

(1) $f_*: \pi_1(A) \longrightarrow \pi_1(M)$ *is monic.*

(2) $f(c_1) \subset F$.

(3) $f(c_1) \cap f(c_2)$ *is empty.*

(4) $[f(c_1)]$ *is not conjugate to a power of* τ_i *for* $1 \leqslant i \leqslant n$.

Then we can find an embedding $g: (A, \partial A) \longrightarrow (M, \partial M)$ *such that*

(1) $g_*: \pi_1(A) \longrightarrow \pi_1(M)$ *is monic.*

(2) $g(c_j)$ *lies in any prespecified neighborhood of* $f(c_j)$ *for* $j = 1, 2$.

(3) $[g(c_1)]$ *is not conjugate to a power of* τ_i *for* $i = 1, \ldots, n$.

PROOF. The proof of Theorem 3 varies very little from the standard one involving a tower of 2-sheeted coverings. Suppose that $f_m: (A, \partial A) \longrightarrow (M_m, \partial M_m)$ is a map such that $f_{m*}(\pi_1(A))$ is not contained in any subgroup of index two of $\pi_1(M_m)$, $q: M_m \longrightarrow M$ is a map such that $qf_m = f$, and $f_m(A)$ is a deformation retract of M_m. The usual homology arguments show that ∂M_m is the union of a torus T and a collection of 2-spheres. Since f_* is monic, $f(\partial A) \subset T$. Let R be a regular neighborhood of $f_m(c_1)$ in T. Let A_2 be the closure of the component of

$T - R$ which contains $f_m(c_2)$. Since $qf_m(c_2)$ is essential and ∂A_2 can contain at most two components, A_2 is an annulus. Let A_1 be the closure of $T - A_2$. Let λ_j be a simple loop in $f_m(c_j)$ such that λ_j represents a generator of $\pi_1(A_j)$ for $j = 1, 2$. Now $[f_m(c_j)] = [\lambda_j]^v \in \pi_1(A_j, x)$ where $x \in \lambda_j$ and v is an integer. It follows that $qf_m(c_1) = f(c_1)$ is homotopic to a nonzero multiple of $q(\lambda_1)$ in F since F is incompressible. Thus if $[f(c_1)]$ is not conjugate to a power of τ_i, $[q(\lambda_1)]$ will not be conjugate to a power of τ_i for $i = 1, \ldots, n$. Let $g_m: A \longrightarrow \partial M_m$ be an embedding such that $g_m(\partial A) = \lambda_1 \cup \lambda_2$. It can now be seen that we may replace the map $qf_m = f$ with the map qg_m. The argument above shows that the map of A into the manifold at the top of the tower may be taken to be an embedding.

It remains to be shown that we can "push this embedding down the tower." Suppose (M_m, p_m) is a 2-sheeted cover of M_{m-1}, $f_m: (A, \partial A) \longrightarrow (M_m, \partial M_m)$ is an embedding, $f_{m-1} = p_m f_m$, and $q: M_{m-1} \longrightarrow M$ is a map such that $qf_{m-1} = f$. We may suppose that the singular set of f_{m-1} is the union of a collection of pairwise disjoint arcs and loops. Clearly the loops can be disposed of by using the standard argument. Suppose α_1 and α_2 are disjoint spanning arcs on A and that $f_{m-1}(\alpha_1) = f_{m-1}(\alpha_2)$. Let a_1 and a_2 be the closures of the components of $c_1 - (\alpha_1 \cup \alpha_2)$. Let x be the point $f(\partial a_1)$. Now $[f(c_1)] = [f(a_1)]\,[f(a_2)]$ in $\pi_1(F, x)$. If $[f(a_1)]$ (or $[f(a_2)]$) is not of the form $\sigma_1 \tau_i^u \sigma_1^{-1}$ where $\sigma_1 \in \pi_1(F, x)$ and u is a nonzero integer, we can find an embedding $g_m: A \longrightarrow M_m$ such that $qp_m g_m(c_1) = f(a_1)$ (or $f(a_2)$). Otherwise $[f(a_1)] = \sigma_1 \tau_i^u \sigma_1^{-1}$ and $[f(a_2)] = \sigma_2 \tau_j^v \sigma_2^{-1}$ where i, j, u, v are nonzero integers such that $1 \leqslant i, j \leqslant n$ and $\sigma_1, \sigma_2 \in \pi_1(F, x)$. Since $\tau_i \neq \tau_j$ or $\sigma_1 \neq \sigma_2$, $\sigma_1 \tau_i^u \sigma_1^{-1} \sigma_2 \tau_j^{-v} \sigma_2^{-1} \neq \sigma_3 \tau_k^w \sigma_3^{-1}$ for $\sigma_3 \in \pi_1(F, x)$, $1 \leqslant k \leqslant n$, and w a nonzero integer. Thus it can be seen that the usual cutting argument will suffice.

Suppose α_1 and α_2 are disjoint simple arcs on A such that $f_{m-1}(\alpha_1) = f_{m-1}(\alpha_2)$ and $\partial\alpha_j \subset c_1$ for $j = 1, 2$. Then α_1 and α_2 cut off disks $\mathcal{D}_1$ and $\mathcal{D}_2$ on A. Let β_j be the closure of $(\partial \mathcal{D}_j) - \alpha_j$ for $j = 1, 2$. We observe that the loop $f_{m-1}(\beta_1) \cup f_{m-1}(\beta_2)$ is nullhomotopic on the singular disk $f_{m-1}(\mathcal{D}_1 \cup \mathcal{D}_2)$. Thus the loop $f(\beta_1) \cup f(\beta_2)$ is nullhomotopic in M and thus on F since F is incompressible. It follows that one could have simplified the singular set of $f|c_1$ via a homotopy or that one can use the usual cutting argument.

It can be seen that if α_1 and α_2 are disjoint simple arcs properly embedded in A such that $\partial\alpha_j \subset c_2$ for $j = 1, 2$ and $f_{m-1}(\alpha_1) = f_{m-1}(\alpha_2)$, the standard cutting argument yields a map agreeing with f on c_1. Theorem 3 follows.

V. Supporting lemmas. In this section we prove the lemmas which are necessary in our proof of Waldhausen's "Torus theorem". It is a consequence of Lemma 5.1 below that an essential loop λ on a good surface F embedded in a

3-manifold M is not freely homotopic to a loop in ∂M unless λ is freely homotopic in F to a loop in ∂F or F is the planar surface with three boundary components.

LEMMA 5.1. *Let F be an incompressible, boundary incompressible, nonseparating surface properly embedded in M such that F is not the planar surface with* $\chi(F) = -1$. *Let* $f\colon A \longrightarrow M$ *be a map such that* $f(c_1) \subset F$, $f(c_1)$ *is essential,* $f(c_2) \subset \partial M$, *and* $f(c_1)$ *is not freely homotopic in F to a loop in* ∂F. *Then F is not good.*

PROOF. After a general position argument, we may suppose that there is an annular neighborhood A_1 of c_1 in A such that $f^{-1}(F) \cap A_1 = c_1$ since F is two-sided. We may also assume that $f^{-1}(F)$ is the union of a finite collection of disjoint simple arcs and loops properly embedded in A. We suppose that f has been chosen so that the number of components in $f^{-1}(F)$ is minimal.

Let λ be a simple loop in $f^{-1}(F) - c_1$. Suppose λ is nullhomotopic on A. Since F is incompressible, $f(\lambda)$ is nullhomotopic on F. Let $\mathcal{D}$ be a regular neighborhood of the disk bounded by λ in A. Then we can alter f on $\mathcal{D}$ to obtain a map $f_1\colon A \longrightarrow M$ such that $f_1|A - \mathcal{D} = f|A - \mathcal{D}$ and $f_1(\mathcal{D}) \cap F$ is empty. This contradicts our assumption of minimality on $f^{-1}(F)$.

Suppose λ is not nullhomotopic on A and $f(\lambda)$ is not freely homotopic on F to a loop in ∂F. Let A_1 be the subannulus of A bounded by c_2 and λ. Then the existence of the map $f_1 = f|A_1\colon A_1 \longrightarrow M$ contradicts our minimality hypothesis since $f_1^{-1}(F)$ contains fewer components than $f^{-1}(F)$. Thus we may suppose that $f(\lambda)$ is freely homotopic to a loop in ∂F. Let A_1 be a regular neighborhood of the annulus bounded by c_2 and λ. Then we can alter f on A_1 to obtain a map $f_1\colon (A, c_2) \longrightarrow (M, \partial M)$ such that $f_1|A - A_1 = f|A - A_1$ and $f(A_1)$ does not meet F. This contradicts our minimality hypothesis above. Thus we may suppose that $f^{-1}(F) - c_1$ contains no simple loops.

Suppose there is an arc α in $f^{-1}(F) - c_1$. Then we can choose α so that there is a disk $\mathcal{D}$ on A such that $\mathcal{D} \cap f^{-1}(F) = \alpha$ and $\partial\mathcal{D} = \alpha \cup (\mathcal{D} \cap f^{-1}(\partial M))$. It is a consequence of Lemma 3.1, that we can alter f on $\mathcal{D}$ to obtain a map $f_1\colon A \longrightarrow M$ such that $f_1|A - \mathcal{D} = f|A - \mathcal{D}$ and $f_1(\mathcal{D}) \subset F$. One can now modify f_1 on a regular neighborhood of $\mathcal{D}$ to obtain a map f_2 such that the number of components in $f_2^{-1}(F)$ is less than the number of components in $f^{-1}(F)$. This contradicts our minimality hypothesis. Thus we may suppose that $f^{-1}(F) = c_1$.

It is a consequence of Theorem 3 that there is an embedding $g\colon A \longrightarrow M$ such that

(1) $g(c_1) \subset F$ and $g(c_2) \subset \partial M$.

(2) $g^{-1}(F) = c_1$.

(3) $g(c_1)$ is not freely homotopic on F to a loop in ∂M.

(4) $g_*: \pi_1(A) \longrightarrow \pi_1(M)$ is monic.

Let R be a regular neighborhood of $F \cup g(A)$ and $G = \text{cl}(\partial R - \partial M)$. Let F_1 be the component of G parallel to F. Now $\chi(G) = 2 * \chi(F)$. If $G - F_1$ is connected, genus$(G - F_1) <$ genus(F). Otherwise let F_2 and F_3 be the components of G. Then $\chi(F_2) > \chi(F)$ and $\chi(F_3) > \chi(F)$. Since F is incompressible it can be seen that $F_2 \cup F_3$ is incompressible. Since $[F_1]$ is not a boundary rel ∂M and $[G - F_1]$ is homologous to F_1 rel ∂M, $[G - F_1]$ is not a boundary rel ∂M. Let F_2 be a component of $G - F_1$. Then we may suppose that F_2 does not separate M. If F_2 is boundary incompressible, it can be seen that F is not good. Otherwise there is a disk $\mathcal{D}$ embedded in M such that $\mathcal{D} \cap F_2$ is an arc β and $\mathcal{D} \cap (\partial M \cup F_2) = \partial \mathcal{D}$. Using an argument similar to the one above we can find a nonseparating, incompressible surface F_2' properly embedded in M such that $\chi(F_2') > \chi(F_2)$. Since $\chi(F_2') \leqslant 1$, it can be seen that F_2' may be taken to be boundary incompressible and thus F is not good. This completes the proof of Lemma 5.1.

Lemma 5.2 below is proved using techniques similar to those used in the proof of Lemma 5.1 above and we leave most of the details to the reader.

LEMMA 5.2. *Let F be a good surface properly embedded in M and R a regular neighborhood of F in M. Let $\pi_2(M) = 0$. Let $g: (A, \partial A) \longrightarrow (M, \text{int}(F))$ be an F-essential embedding such that*

(1) *$g(A)$ meets only one component of $R - F$.*

(2) *$g(c_1)$ is not freely homotopic in F to a loop in ∂F.*

(3) *$g_*: \pi_1(A) \longrightarrow \pi_1(M)$ is monic.*

(4) *$g^{-1}(F) = \partial A$.*

Let N be a regular neighborhood of $F \cup g(A)$. Let F_1 be the nonseparating component of cl$(\partial N - \partial M)$ *not parallel to F. Then F_1 is good.*

PROOF. We observe that cl$(\partial N - \partial M)$ has at most three components and that one of them say F_1 must not separate M. Note that $\chi(F_1) \leqslant \chi(F)$. If F_1 is not incompressible there is a disk $\mathcal{D}$ embedded in M such that $\mathcal{D} \cap F_1 = \partial \mathcal{D}$. Using $\mathcal{D}$ and F_1, we can find a surface F_2 properly embedded in M that does not separate M so that $\chi(F_2) > \chi(F_1) \geqslant \chi(F)$. Since $\pi_2(M) = 0$, F_2 cannot be a 2-sphere. As in the proof of Lemma 5.1, the existence of F_2 guarantees the existence of a good surface F_3 such that $\chi(F_3) > \chi(F)$. This is a contradiction. Thus F_1 is incompressible. Similar arguments show F_1 is boundary incompressible. Lemma 5.2 follows.

A connected, closed 2-manifold K that is the union of a collection of annuli A_i for $i = 1, \ldots, n$ such that each component of $A_i \cap A_j$ is a boundary component of A_i and of A_j for $1 \leqslant i < j \leqslant n$ is known to be either a Klein bottle or a torus. In Lemma 5.3 below we show that a map $f: K \longrightarrow M$ constructed in a particular way is essential.

LEMMA 5.3. *Let F be an incompressible surface properly embedded in a 3-manifold M such that $\pi_2(M) = 0$. Let K be the union of a collection $A_1, \ldots, A_n$ of annuli such that each component of $A_i \cap A_j$ is a boundary component of A_i and A_j for $1 \leqslant i < j \leqslant n$. Suppose $f: K \longrightarrow M$ is a map such that*

(1) $f^{-1}(F) = \bigcup_{i=1}^n \partial A_i$.

(2) $f|A_i: (A_i, \partial A_i) \longrightarrow (M, F)$ *is not homotopic* rel ∂A_i *to a map into F for $i = 1, \ldots, n$.*

(3) *There is an essential simple loop λ_1 in $K - \bigcup_{i=1}^n \partial A_i$ such that $f(\lambda_1)$ is essential in M.*

Then $f_*: \pi_1(K) \longrightarrow \pi_1(M)$ is monic.

PROOF. Let λ_2 be a simple loop on K that meets each component of $\bigcup_{i=1}^n \partial A_i \cup \lambda_1$ in a single point and crosses that component at the point of intersection. Now each element $\sigma \in \pi_1(K)$ is represented by a loop on $\lambda_1 \cup \lambda_2$. Suppose $\sigma \in \pi_1(K)$ and $f_*(\sigma) = 1$. Then there is a map $\phi: S^1 \longrightarrow K$ such that

(1) $[\phi(S^1)] = \sigma$.

(2) $\phi(S^1) \subset \lambda_1 \cup \lambda_2$.

(3) $\phi(S^1)$ is not homotopic in K to a map $\phi_1: S^1 \longrightarrow \lambda_1 \cup \lambda_2$ such that $\phi_1^{-1}(\bigcup_{i=1}^n \partial A_i)$ contains fewer points than $\phi^{-1}(\bigcup_{i=1}^n \partial A_i)$.

If the cardinality of $\phi^{-1}(\bigcup_{i=1}^n \partial A_i)$ is zero, $\phi(S^1)$ is homotopic to a multiple of λ_1; and since $f(\lambda_1)$ is essential, $[\phi(S^1)] = 1 \in \pi_1(K)$. Otherwise $f\phi: S^1 \longrightarrow M$ is an essential map.

Let $\mathcal{D}$ be the unit disk so $\partial \mathcal{D} = S^1$ and $\Phi: \mathcal{D} \longrightarrow M$ a map such that $\Phi|\partial\mathcal{D} = f\phi$. After the usual argument we may suppose that $\Phi^{-1}(F)$ is the union of a collection of arcs properly embedded in $\mathcal{D}$ since F is incompressible. But we can find $\beta_1 \subset \Phi^{-1}(F)$ such that β_1 cuts off a disk $\mathcal{D}_1$ on $\mathcal{D}$ so that $\mathcal{D}_1 \cap \Phi^{-1}(F) = \beta_1$. Let β_2 be $\partial\mathcal{D}_1 - \beta_1$. Now $\phi(\beta_2)$ is homotopic to a spanning arc of A_j where j is some integer such that $1 \leqslant j \leqslant n$. It can be seen that $f\phi(\beta_2)$ is homotopic rel its endpoints to $\Phi(\beta_1)$ across $\Phi(\mathcal{D}_1)$. It is a consequence of Proposition 3.3 that $f|A_j$ is homotopic rel ∂A_j to a map into F. This contradicts our hypotheses on f and we conclude $\sigma = 1$ and f_* is monic. This completes the proof of Lemma 5.3.

LEMMA 5.4. *Let M be a 3-manifold such that $\pi_2(M) = 0$ and ∂M is incompressible and let $f: T \longrightarrow M$ be a W-essential map. Then there is an essential embedding $g: (A, \partial A) \longrightarrow (M, \partial M)$.*

PROOF. Since f is W-essential there is an element $\sigma \in \pi_1(T)$ such that $f_*(\sigma)$ has a representative loop l that is freely homotopic to a loop in ∂M. Let λ_1 and λ_2 be simple loops on T that meet in a single point and cross at that point. Now $\sigma = k_1[\lambda_1] + k_2[\lambda_2]$ where k_1 and k_2 are integers. Let $(\widetilde{T}, q)$ be the finite sheeted covering space of T associated with the subgroup of $\pi_1(T)$ generated by $[\lambda_1]$ and $k_2[\lambda_2]$. Then $(fq)_*: \pi_1(\tilde{T}) \longrightarrow \pi_1(M)$ is one-one and there is a simple loop μ on $\tilde{T}$ such that $[q\mu] = \sigma$. Since l is homotopic to a loop in ∂M, fq is homotopic to a map $f_1: \tilde{T} \longrightarrow M$ such that $f_1(\mu) \subset \partial M$. Now f_1 induces a map $\bar{f}_1: (A, \partial A) \longrightarrow (M, f_1(\mu))$ in a natural way. If $\bar{f}_1$ is not an essential map, $\bar{f}_1: (A, \partial A) \longrightarrow (M, \partial M)$ is homotopic rel ∂A to a map into ∂M by Proposition 3.3. It follows that each representative of $f_*[\lambda_1]$ is freely homotopic to a loop in ∂M. Thus we may suppose that $f(\lambda_1) \subset \partial M$. But then f induces a map $f_2: (A, \partial A) \longrightarrow (M, \partial M)$ in a natural way and f_2 must be essential since f was not homotopic to a map into ∂M. An application of Theorem 2 completes the proof of Lemma 5.4.

Let F be a good surface in M. The next three lemmas will be used in the proof of the torus theorem in showing that if $f: T \longrightarrow M$ is an essential map, we may suppose that f is not homotopic to a map f_1 such that $f_1^{-1}(F)$ is empty.

LEMMA 5.5. *Let M_1 be a submanifold of the 3-manifold M and ∂M_1 incompressible in M. Let $g: T \longrightarrow M_1$ be an essential embedding. Then $g(T)$ is essential in M.*

PROOF. Since ∂M_1 is incompressible, it is a consequence of standard geometric arguments that $\pi_1(M_1) \longrightarrow \pi_1(M)$ is monic. Thus $g(T)$ is incompressible in M. Let λ be a loop on $g(T)$. Suppose $f: A \longrightarrow M$ is a free homotopy of λ to a loop in ∂M. After the usual argument, we may suppose that $f^{-1}(\partial M_1)$ is the union of a collection of disjoint, simple essential loops. But then λ is freely homotopic to a loop in ∂M_1. It follows that g is an essential map. This completes the proof of Lemma 5.5.

LEMMA 5.6. *Let M be an irreducible 3-manifold. Let $g: (A, \partial A) \longrightarrow (M, \partial M)$ be an essential embedding. Let $f: T \longrightarrow M$ be an essential map such that there is no map f_1 homotopic to f so that $f_1^{-1}(g(A))$ is empty. Then M admits an embedding $h: T \longrightarrow M$ such that h_* is monic and h is not homotopic to a map h_1 such that $h_1(T) \cap g(A)$ is empty. In fact $h^{-1}g(A)$ is the union of one, two, or four disjoint simple essential loops.*

PROOF. We may assume that $f^{-1}(g(A))$ is a collection of disjoint simple loops after a general position argument. We suppose that f has been chosen so that $f^{-1}(g(A))$ contains as few loops as possible. Suppose some loop λ in $f^{-1}(g(A))$ is inessential. Then $f(\lambda)$ is nullhomotopic on $g(A)$. Let $\mathcal{D}$ be a regular neighborhood of the disk on T bounded by λ. Then we can find a map $f_1: T \longrightarrow M$ such that $f_1 | T - \mathcal{D} = f | T - \mathcal{D}$ and $f_1(\mathcal{D}) \cap g(A)$ is empty. Since $\pi_2(M) = 0$, f and f_1 are homotopic. Thus the existence of f_1 contradicts our choice of f and we suppose that each loop in $f^{-1}g(A)$ is essential.

Let $A_1, \ldots, A_n$ be the closures of the components of $T - f^{-1}g(A)$. Let R be a regular neighborhood of $g(A)$. Let M_1 be the manifold obtained by splitting M along $g(A)$ and P: $M_1 \longrightarrow M$ the natural map.

Suppose that $f(A_1)$ meets both components of $R - g(A)$. We apply the theorem in [14] to find an embedding h_1: $(A, \partial A) \longrightarrow (M_1, P^{-1}g(A))$ such that $h_1(\partial A)$ meets both components of $P^{-1}(g(A))$ and h_{1*} is monic. We may suppose that $Ph_1(c_1) = Ph_1(c_2)$. Then Ph_1 defines an embedding $\bar{h}$: $K \longrightarrow M$ where K is a closed surface of genus 1. Since $h_1|\alpha$ is not homotopic rel $\partial\alpha$ to a map into $P^{-1}g(A)$, it follows from the argument given in the proof of Lemma 5.3 that $\bar{h}_*$: $\pi_1(K) \longrightarrow \pi_1(M)$ is monic. If K is a torus we let $h = \bar{h}$. Otherwise the boundary of a regular neighborhood N of $\bar{h}(K)$ is a torus embedded in M since M is orientable and ∂N must be orientable. Since $\bar{h}_*$ is monic, $\pi_1(\partial N) \longrightarrow \pi_1(M)$ is monic and any embedding h: $T \longrightarrow \partial N$ has the desired properties. Thus we may suppose that $f(A_i)$ meets a single component of $R - g(A)$ for $i = 1, \ldots, n$.

If $n = 1$, f is homotopic to a map f_1 such that $f_1(T) \cap g(A)$ is empty; so $n \geqslant 2$. We may suppose that $f(A_1 \cup A_2)$ meets both components of $R - g(A)$. Let M_1 and P be as above. Let h_j: $(A_j, \partial A_j) \longrightarrow (M_1, P^{-1}g(A))$ be the maps induced by f for $j = 1, 2$. Then h_j is not homotopic rel ∂A_j to a map into $P^{-1}g(A)$ for $j = 1, 2$. Let α_j be a spanning arc of A_j for $j = 1, 2$. It is a consequence of Theorem 2 that there exist embeddings g_j: $(A_j, \partial A_j) \longrightarrow (M_1, P^{-1}(g(A)))$ that are $P^{-1}g(A)$-essential and $g_j(A_j)$ meets the same component of $P^{-1}(g(A))$ as does $h_j(A_j)$ for $j = 1, 2$. We suppose that g_1 and g_2 are in general position with respect to one another. Then $g_1(A_1) \cap g_2(A_2)$ is a collection of simple loops. We can remove any inessential loops by surgery. If any loops remain we can find an embedding $\bar{g}$: $(A, \partial A) \longrightarrow (M_1, P^{-1}g(A))$ such that $\bar{g}_*$ is monic and $\bar{g}(\partial A)$ meets both components of $P^{-1}g(A)$. In this case we obtain the desired result as above. Thus we may suppose that $g_1(A_1) \cap g_2(A_2)$ is empty and that $Pg_1(\partial A_1) = Pg_2(\partial A_2)$. Now $Pg_1(A_1) \cup Pg_2(A_2)$ induces an embedding $\bar{h}$: $K \longrightarrow M$ where K is a surface of genus 1. Since g_j is $P^{-1}g(A)$-essential for $j = 1, 2$, $\bar{h}_*$: $\pi_1(K) \longrightarrow \pi_1(M)$ is monic by Proposition 3.2 and Lemma 5.3. One now finds the embedding h: $T \longrightarrow M$ as above. This completes the proof of Lemma 5.6.

LEMMA 5.7. *Let f: $T \longrightarrow M_1$ be an essential map. Let M_j, $F_j \subseteq M_j$, $U(F_j) \subseteq M_j$ be a special hierarchy of M_1 with respect to f. Let k be the smallest integer such that for every map f_1 homotopic to f, if $f_1(T) \subset M_k$, $f_1(T) \cap F_k$ is not empty. If M_k admits an essential embedding g of A or T, M_1 admits an essential embedding of A or T.*

PROOF. Suppose g: $T \longrightarrow M_k$ is an essential embedding. Since F_k is not a disk and ∂M_k is incompressible in M_1, it is a consequence of 5.5 that $g(T)$ is essential in M_1. Thus we may suppose that $g: (A, \partial A) \longrightarrow (M_k, \partial M_k)$ is an essential embedding. We may suppose $f(T) \subset M_k$. Since $g(A)$ could have been chosen as F_k, $g(A) \cap f(T)$ is not empty. We may suppose that $f^{-1}(g(A))$ is a collection of simple essential loops.

Suppose λ is one of these loops and $f(\lambda)$ is freely homotopic to a loop in ∂M_1. Then $f: T \longrightarrow M_1$ is W-essential and M_1 admits an essential embedding of an annulus by Lemma 5.4. Otherwise, we can find an embedding $h: T \longrightarrow M_k$ such that h_* is monic and $h(\lambda) \subset g(A)$ is freely homotopic to $g(c_1)$ in $g(A)$ by Lemma 5.6. Note that $h(\lambda)$ is not freely homotopic to a loop on ∂M_1 since $f(\lambda)$ is a multiple of $h(\lambda)$. It follows that $h: T \longrightarrow M_1$ is essential. This completes the proof of Lemma 5.7.

LEMMA 5.8. *Let F be a surface properly embedded in M. Let $f: (A, \partial A) \longrightarrow (M, \partial M)$ be a map which is transverse with respect to F such that $f^{-1}(F)$ is a collection of simple loops essential in A. Then we may assume that any embedding $g: (A, \partial A) \longrightarrow (M, \partial M)$ constructed via a tower argument from f meets F in a collection of disjoint simple loops and that there is a fixed finite number of possibilities for $g(A) \cap F$ up to ambient isotopy in F. Furthermore the number of these possibilities is determined by the set $ff^{-1}(F)$.*

PROOF. The proof of Lemma 5.8 is similar to that of Lemma 4.6 in [5] and we omit it.

LEMMA 5.9. *Let F be an incompressible surface in ∂M and F_1 a planar submanifold of F such that $\pi_1(F_1) \longrightarrow \pi_1(F)$ is monic and $\chi(F_1) = -1$. Let $f: (A, \partial A) \longrightarrow (M, F)$ be an F-essential map such that*

(1) *$f(c_1) \subset F_1$ and $f(c_2) \subset F - \partial F_1$.*

(2) *$f(c_1)$ is not freely homotopic on F to a multiple of a loop in ∂F_1. Let λ be a component of ∂F_1. Then there is an F-essential embedding $g: (A, \partial A) \longrightarrow (M, F)$ such that*

(1) *$g(c_1) = \lambda$.*

(2) *$g(c_2)$ lies in any prespecified neighborhood of $f(c_2)$.*

PROOF. Let (M^*, p) be the covering space of M associated with $\pi_1(F) \subset \pi_1(M)$. Let $\bar{F}$ be a component of $p^{-1}(F)$ such that $p|\bar{F}$ is a homeomorphism. Let $\bar{f}_1: (A, \partial A) \longrightarrow (M^*, \partial M^*)$ be a map such that $p\bar{f}_1 = f$ and $\bar{f}_1(c_1) \subset \bar{F}_1 = p^{-1}(F_1) \cap \bar{F}$. Since $f(\alpha)$ is not homotopic rel its boundary to an arc on F and $\pi_1(\bar{F}) \longrightarrow \pi_1(M^*)$ is an isomorphism, $\bar{f}_1(c_2)$ does not lie on $\bar{F}$. It is a consequence of Theorem 3 that there is an embedding $\bar{g}_1: (A, \partial A) \longrightarrow (M^*, \partial M^*)$ such that

(1) $\bar{g}_1(c_1)$ is freely homotopic in $\bar{F}$ to $\bar{F} \cap p^{-1}(\lambda)$.

(2) $\bar{g}_1(c_2)$ lies in any prespecified neighborhood of $\bar{f}_1(c_2)$.

After a homotopy, we may suppose that $\bar{g}_1(c_1) = \bar{F} \cap p^{-1}(\lambda)$. Let $g_1 = p\bar{g}_1$.

Then $g_1(c_1) = \lambda$ and $g_1(c_2)$ lies in a neighborhood of $f(c_2)$. Now the arc $g_1(\alpha)$ is not homotopic to an arc on F since the components of $\bar{g}_1(\partial\alpha)$ lie on distinct components of $p^{-1}(F)$. Lemma 5.9 can now be seen to be a consequence of Theorem 2.

LEMMA 5.10. *Let M be an irreducible 3-manifold with incompressible boundary. Let F be an incompressible, boundary incompressible planar surface properly embedded in M such that $\chi(F) = -1$. Let $f\colon T \longrightarrow M$ be an essential map such that*

(1) *$f^{-1}(F)$ is a nonempty collection of disjoint simple essential loops.*

(2) *f is not homotopic to a map $f_1\colon T \longrightarrow M$ such that $f_1^{-1}(F)$ contains fewer components than $f^{-1}(F)$ and $f_1^{-1}(F)$ is the union of a collection of disjoint simple loops.*

Then there is an essential embedding $g\colon (A, \partial A) \longrightarrow (M, \partial M)$.

PROOF. If f is W-essential, Lemma 5.10 is a direct consequence of 5.4. Otherwise let $\lambda_1, \ldots, \lambda_n$ be the components of $f^{-1}(F)$. Since f is S-essential, $f(\lambda_i)$ is not freely homotopic in F to a loop in ∂F. We suppose that the closures of the components of $T - \bigcup_{i=1}^n \lambda_i$ are denoted by $A_1, \ldots, A_n$.

Let R be a regular neighborhood of F. We divide the proof of Lemma 5.10 into two cases:

Case 1. $f(A_1)$ meets both components of $R - F$.

Case 2. $f(A_1)$ meets only one component of $R - F$.

We now proceed with the proof of Case 1.

Let M_1 be the 3-manifold obtained by splitting M along F, $\bar{f}\colon A \longrightarrow M_1$ be the map induced by $f|A_1$ and $P\colon M_1 \longrightarrow M$ the natural projection map from M_1 onto M. Let F_j be the component of $P^{-1}(F)$ such that $\bar{f}(c_j) \subset F_j$ for $j = 1, 2$. Note $F_1 \neq F_2$. Let $(\widetilde{M}_1, p)$ be the covering space of M_1 associated with $\pi_1(F_1) \subset \pi_1(M_1)$. We may suppose that $\widetilde{f}\colon A \longrightarrow \widetilde{M}_1$ is a map such that $p\widetilde{f} = \bar{f}$. Let $\widetilde{F}_j$ be the component of $p^{-1}(F_j)$ on which $\widetilde{f}(c_j)$ lies for $j = 1, 2$. We may assume that $p|\widetilde{F}_1$ is a homeomorphism. Let G be the component of $\partial\widetilde{M}_1$ which contains $\widetilde{F}_1$.

We claim that if every component of $\partial\widetilde{F}_1$ is freely homotopic in G to a loop in $\widetilde{F}_2$, $\widetilde{M}_1$ is homeomorphic to $F \times [0, 1]$.

Let l_1, l_2, and l_3 be the components of $\partial\widetilde{F}_1$. Suppose $h\colon l_1 \times [0, 1] \longrightarrow G$ is a free homotopy of l_1 to a loop in $\widetilde{F}_2$. We may suppose that $h^{-1}\partial(\widetilde{F}_1 \cup \widetilde{F}_2)$ is a collection of disjoint simple loops $\mu_1, \ldots, \mu_m$ and $h(\mu_1) = l_1$. We may assume that h has been chosen so that m is minimal. Then μ_i is an essential loop in $l_1 \times [0, 1]$ as is $h(\mu_i)$ in G. We observe that $\partial\widetilde{F}_2$ is composed of disjoint simple loops and a number of copies of the real line. It can be seen that $h(\mu_i)$ must lie on one of the simple loops. Since any arc in $\widetilde{M}_1$ with its boundary on

$\widetilde{F}_1$ is homotopic rel its boundary to an arc on $\widetilde{F}_1$ and since $\pi_2(\widetilde{M}_1) = 0$, one can deform any free homotopy of pairs of loops in $\widetilde{F}_1$ until the free homotopy lies in $\widetilde{F}_1$. Since no pair of boundary components of $\widetilde{F}_1$ are freely homotopic in $\widetilde{F}_1$, no pair of boundary components of $\widetilde{F}_1$ are freely homotopic in G. Thus it can be seen that we may suppose that $h(\mu_i)$ does not meet $\widetilde{F}_1$ for $i = 1, \ldots, m$ and l_1 is freely homotopic to a power of a boundary component $h(\mu_2)$ of $\widetilde{F}_2$. Note $h(\mu_2)$ is an essential loop in $\widetilde{F}_2$ and $l_1 \cup h(\mu_2)$ bounds an annulus embedded in G.

Suppose that l_1, l_2, and l_3 are freely homotopic in G to loops in $\widetilde{F}_2$. Then we can find disjoint simple loops μ_1, μ_2, and μ_3 in $\partial\widetilde{F}_2$ such that $\mu_j \cup l_j$ bounds an annulus $\bar{A}_j$ embedded in G for $j = 1, 2, 3$. Since $\widetilde{F}_2$ is connected we can find disjoint simple arcs α_1 and α_2 properly embedded in $\widetilde{F}_2 \cup \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$ such that

(1) $\alpha_j \cap \widetilde{F}_2$ is an arc properly embedded in $\widetilde{F}_2$ for $j = 1, 2$.

(2) α_1 meets both l_1 and l_2.

(3) α_2 meets both l_1 and l_3.

Note that α_1 meets μ_1 and μ_2 and α_2 meets μ_1 and μ_3. We observe that α_1 and α_2 are homotopic rel their boundaries to arcs on $\widetilde{F}_1$ since the natural map $\pi_1(\widetilde{F}_1) \longrightarrow \pi_1(\widetilde{M}_1)$ induced by inclusion is onto. It is a consequence of the proof of the loop theorem [12] that there are disjoint disks $\mathcal{D}_1$ and $\mathcal{D}_2$ properly embedded in $\widetilde{M}_1$ such that $\mathcal{D}_j \cap \widetilde{F}_1$ is a simple arc properly embedded in $\widetilde{F}_1$ and $\mathcal{D}_j \cap (G - \widetilde{F}_1) = \text{int}(\alpha_j)$ for $j = 1, 2$. We observe that $\widetilde{F}_1 - (\mathcal{D}_1 \cup \mathcal{D}_2)$ is simply connected. Let $l \subset \text{int}(\widetilde{F}_2)$ be a boundary component of a regular neighborhood of $\mu_1 \cup \mu_2 \cup \mu_3 \cup ((\alpha_1 \cup \alpha_2) \cap \widetilde{F}_2)$. Now l lies in the complement of $\mathcal{D}_1 \cup \mathcal{D}_2$ and there is a free homotopy of l to a loop in $\widetilde{F}_1$. It can be shown that this free homotopy can be assumed to lie in the complement of $\mathcal{D}_1 \cup \mathcal{D}_2$ using an argument similar to one given above. It follows that l is inessential in $\widetilde{M}_1$. Since F is incompressible in M, $\widetilde{F}_2$ is incompressible in $\widetilde{M}_1$. Thus l is null-homotopic on $\widetilde{F}_2$ and l bounds a disk on $\widetilde{F}_2$. It follows that $\widetilde{F}_2$ is homeomorphic to F. Let B be the manifold obtained by removing the interior of a regular neighborhood of $\mathcal{D}_1 \cup \mathcal{D}_2$ from $\widetilde{M}_1$. Then one component of ∂B is a 2-sphere. Since $\pi_2(\widetilde{M}_1) = 0$, B is a homotopy 3-cell. It can now be seen that $p\colon \widetilde{M}_1 \longrightarrow M_1$ is a homeomorphism and that $\widetilde{M}_1$ is homeomorphic to $F \times I$. This establishes our claim.

We suppose that $\widetilde{M}_1$ is not homeomorphic to $F \times [0, 1]$. As a consequence of our claim above, it follows that we may assume that the loop l_1 in $\partial\widetilde{F}_1$ is not freely homotopic in G to a loop in $\widetilde{F}_2$. It is a consequence of Theorem 3 that the existence of $\widetilde{f}\colon A \longrightarrow \widetilde{M}_1$ guarantees the existence of an embedding $\widetilde{g}_1\colon A \longrightarrow \widetilde{M}_1$ such that

(1) $\widetilde{g}_1(c_j) \subset \widetilde{F}_j$ for $j = 1, 2$.

(2) $\widetilde{g}_1(c_1)$ is freely homotopic to l_1.

(3) $\widetilde{g}_{1*}\colon \pi_1(A) \longrightarrow \pi_1(\widetilde{F}_j)$ is monic.

Since $\widetilde{g}_1(c_1)$ is simple, we may assume that $\widetilde{g}_1(c_1) = l_1$. Since $p|\widetilde{F}_1$ is a homeomorphism, $p\widetilde{g}_1|c_1$ is a homeomorphism.

If $p\widetilde{g}_1(c_2)$ is freely homotopic to a loop in ∂F_2, we assume that $p\widetilde{g}_1(c_2) \subset \partial F_2$ and will show below that $Pp\widetilde{g}_1\colon (A, \partial A) \longrightarrow (M, \partial M)$ is an essential map. In this case Lemma 5.10 is a consequence of Theorem 2.

If $P\widetilde{g}_1(c_2)$ is not freely homotopic to a loop in ∂F_2, we replace $p\widetilde{g}_1$ by another map and proceed as has been indicated above.

Suppose $p\widetilde{g}_1(c_2)$ is not freely homotopic in F_2 to a loop in ∂F_2. Let $(\overline{M}_1, q)$ be the covering space of M_1 associated with $\pi_1(F_2) \subset \pi_1(M_1)$. Let $\bar{g}_1\colon A \longrightarrow \overline{M}_1$ be a map such that $q\bar{g}_1 = p\widetilde{g}_1$. *Let* $\overline{F}_j$ be the component of $q^{-1}(F_j)$ such that $\bar{g}_1(c_j) \subset \overline{F}_j$ for $j = 1, 2$. Let $\overline{G}$ be the component of $\partial\overline{M}_1$ which contains $\overline{F}_2$. We may suppose $q|\overline{F}_2$ is a homeomorphism. The proof of Lemma 5.10 is the same in case $\overline{M}_1$ is homeomorphic to $\overline{F}_2 \times [0, 1]$ as it is in case $\widetilde{M}_1$ is homeomorphic to $\widetilde{F}_1 \times [0, 1]$. It follows from the claim above that we may assume some component μ_1 of $\partial\overline{F}_2$ is not homotopic in $\overline{G}$ to a loop in $\partial\overline{F}_1$. As above by Theorem 3, we can find an embedding $\bar{g}_2\colon A \longrightarrow \overline{M}_1$ such that

(1) $\bar{g}_2(c_2) = \mu_1$.

(2) $\bar{g}_2(c_1)$ lies in a small neighborhood in $\partial\overline{M}_1$ of $\bar{g}_1(c_1)$.

(3) $\bar{g}_{2*}\colon \pi_1(A) \longrightarrow \pi_1(\overline{M}_1)$ is monic.

Since $\bar{g}_1(c_1)$ is simple, its neighborhood may be taken to be an annulus and we assume $\bar{g}_2(c_1) = \bar{g}_1(c_1)$. Now $q|\overline{F}_2$ and $q|\bar{g}_1(c_1)$ are homeomorphisms so $q\bar{g}_2|\partial A$ is an embedding and thus $Pq\bar{g}_2(\partial A) \subset \partial F$. Clearly we may as well suppose $Pp\widetilde{g}_1(\partial A) \subset \partial F$. Let $h = Pp\widetilde{g}_1\colon (A, \partial A) \longrightarrow (M, \partial M)$. We claim h is an essential map.

If not, the arc $h(\alpha)$ is homotopic rel its boundary to an arc in ∂M. Since ∂M is incompressible and $\pi_2(M) = 0$, h is homotopic rel ∂A to a map $h_1\colon A \longrightarrow \partial M$ by Proposition 3.2. Let $H\colon A \times [0, 1] \longrightarrow M$ be a homotopy such that

(1) $H(x, 0) = h(x)$ for $x \in A$.

(2) $H(x, 1) = h_1(x)$ for $x \in A$.

(3) $H(x, t) = h(x)$ for $x \in \partial A$ and $t \in [0, 1]$.

After a general position argument, we may suppose that $H^{-1}(F)$ is an incompressible surface properly embedded in $A \times [0, 1]$. We suppose that h_1 has been chosen so that the number of components in $h_1^{-1}(F)$ is minimal. We observe that $H^{-1}(F)$ must be a disjoint union of disks and annuli. Suppose $\mathcal{D}$ is a disk in $H^{-1}(F)$. Since $h^{-1}(F) = \partial A$, $\mathcal{D} \cap \partial(A \times [0, 1]) \subset A \times \{1\}$. Of course $H(\partial\mathcal{D})$ lies on one component of ∂F. Since F is incompressible, it can be seen that h_1 was not chosen so that the number of components in $h_1^{-1}(\partial F)$ would be minimal. Suppose A_1 is an annulus in $H^{-1}(F) - \partial A \times [0, 1]$. Then both components of

$H(\partial A_1)$ lie on ∂F. Since $H(A_1) \subset F$ and each component of ∂A_1 is an essential loop in $A \times [0, 1]$, $H|A_1$ is homotopic rel ∂A_1 to a map into ∂F. Thus we may suppose that $H(A_1) \subset \partial F$. Note that A_1 divides $A \times [0, 1]$ into two solid tori, and that the restriction of H to the one of them which meets $A \times \{0\}$ defines a homotopy of h to a map h_2. It can be seen that the number of components in $h_2^{-1}(F)$ is one less than that of $h_1^{-1}(F)$. It follows that the best possible choice for h_1 was not made. We conclude that $H^{-1}(F) = \partial A \times [0, 1]$.

We may now consider H as a map into M_1. Since H can be lifted to a map $\widetilde{H}: A \times [0, 1] \longrightarrow \widetilde{M}_1$, we see that $\widetilde{g}_1(c_1)$ is homotopic to $\widetilde{g}_1(c_2)$ in $\partial\widetilde{M}_1$. This contradicts our hypothesis that l_1 was not freely homotopic in G to a loop in $\widetilde{F}_2$ and our claim is established.

In this case Lemma 5.10 follows from Theorem 2 above.

Suppose that $\widetilde{M}_1$ is homeomorphic to $F \times [0, 1]$. Since the natural map from $\pi_1(\widetilde{F}_1)$ into $\pi_1(\widetilde{M}_1)$ induced by inclusion is an isomorphism and $p: \widetilde{M}_1 \longrightarrow M_1$ is a homeomorphism, it is a consequence of Theorem 3.1 in [1] that there is a homeomorphism $\theta: F \times [0, 1] \longrightarrow M_1$ such that

(1) $\theta(F \times [0, 1]) = M_1$.

(2) $\theta(F \times \{0\}) = F_1$.

(3) $\theta(F \times \{1\}) = F_2$.

Now $h_j = P|F_j$ is a homeomorphism for $j = 1, 2$. Let $\rho_1: F \times \{0\} \longrightarrow F \times \{1\}$ be the homeomorphism defined by $\rho_1 = \theta^{-1}h_2^{-1}h_1\theta$. Let $\rho: F \longrightarrow F$ be the homeomorphism defined by $\rho(x) = y$ if $\rho_1(x, 0) = (y, 1)$. Let β_0 be a simple separating arc properly embedded in F. We require that neither component of the complement of β_0 is simply connected. Let $\beta_n = \rho^n\beta_0$ for n a positive integer. Then it can be seen that there exists an integer k such that β_0 and β_k are homotopic under a map H: $[0, 1] \times [0, 1]$ such that

(1) $H([0, 1] \times \{0\}) = \beta_0$.

(2) $H([0, 1] \times \{1\}) = \beta_k$.

(3) $H(\{0, 1\} \times [0, 1]) \subset \partial F$.

One then defines an essential map g_0: $(A, \partial A) \longrightarrow (M, \partial M)$ as follows: Let $\mathcal{D}_i = \beta_i \times [0, 1]$ for $i = 0, \ldots, k-1$. Let $\mathcal{D}_k = [0, 1] \times [0, 1]$. Note that $P\theta(\bigcup_{i=0}^{k-1} \mathcal{D}_i)$ together with $H(\mathcal{D}_k) \subset F$ defines a map g_0: $(A, \partial A) \longrightarrow (M, \partial M)$ in a natural way. Furthermore this map is essential since it is a consequence of Lemma 3.1 that $P\theta(\beta_0 \times \{0\})$ is not homotopic rel its boundary to an arc in ∂M for F is boundary incompressible in M.

We may now suppose that $f(A_1)$ meets only one component of $R - F$. We again let M_1 be the manifold obtained by splitting M along F. We define P, F_1, and F_2 as above. Now $f|A_1$ induces a map $\bar{f}: A \longrightarrow M_1$ and we may suppose that $\bar{f}(\partial A) \subset F_1$. We observe that it is a consequence of Proposition 3.2 that

$\bar{f}(\alpha)$ is not homotopic rel $\bar{f}(\partial\alpha)$ to an arc in F_1 since $\pi_2(M) = 0$ and $f|A_1$ is not homotopic rel ∂A_1 to a map into F. Let $(\widetilde{M}_1, p)$ be the covering space of M_1 associated with $\pi_1(F_1) \subset \pi_1(M_1)$. Let $\widetilde{F}_1$ be a component of $p^{-1}(F_1)$ such that $p|\widetilde{F}_1$ is a homeomorphism. Then there is a map $\tilde{f}$: $A \longrightarrow \widetilde{M}_1$ such that $p\tilde{f} = \bar{f}$ and $\tilde{f}(c_1) \subset \widetilde{F}_1$. Let $\widetilde{F}_2$ be the component of $p^{-1}(F_1)$ on which $\tilde{f}(c_2)$ lies. Note that if $\widetilde{F}_1 = \widetilde{F}_2$, $\tilde{f}(\alpha)$ is homotopic rel its boundary to an arc in $\widetilde{F}_1$ and by Proposition 3.2, $\tilde{f}$ is homotopic rel ∂A to a map into $\widetilde{F}_1$. This would contradict condition (2) in the statement of the lemma.

As above we may suppose that some component l_1 of $\partial\widetilde{F}_1$ is not freely homotopic in $\partial\widetilde{M}_1$ to a loop in $\widetilde{F}_2$ for otherwise $\widetilde{M}_1$ is homeomorphic to $\widetilde{F}_1 \times [0, 1]$. We can apply Theorem 3 to find an embedding $\tilde{f}_1$: $A \longrightarrow \widetilde{M}_1$ such that $\tilde{f}_1(c_1) = l_1$ and $\tilde{f}_1(c_2) \subset \widetilde{F}_2$. Thus $p\tilde{f}_1(c_1) \subset \partial F_1$. If $p\tilde{f}_1|c_2$: $c_2 \longrightarrow F_1$ is not an embedding, we reverse the roles of c_1 and c_2 as above to obtain an embedding $\tilde{f}_2$: $A \longrightarrow \widetilde{M}_1$ such that

(1) $p\tilde{f}_2|c_j$ is an embedding for $j = 1, 2$.

(2) $p\tilde{f}_2(\partial A) \subset \partial F_1$.

(3) $\tilde{f}_2(c_2) \subset \widetilde{F}_1$.

(4) $\tilde{f}_2(c_1)$ does not lie on $\widetilde{F}_1$.

(5) $\tilde{f}_2(c_1)$ is not freely homotopic to $\tilde{f}_2(c_2)$ in $\partial\widetilde{M}_1$.

Let $h = Pp\tilde{f}_2$. We claim that h is an essential map. Suppose that $h(\alpha)$ is homotopic rel its boundary to an arc in ∂M. Since M has incompressible boundary and $\pi_2(M) = 0$, there is a map H: $A \times [0, 1] \longrightarrow M$ such that

(1) $H(x, 0) = h(x)$ for $x \in A$.

(2) $H(A \times \{1\}) \subset \partial M$.

(3) $H(x, t) = h(x)$ for $x \in \partial A$ and $t \in [0, 1]$.

We suppose that H has been chosen so that $H^{-1}(F)$ is a system of incompressible surfaces in $A \times [0, 1]$ and $H^{-1}(F)$ contains as few components as possible. One sees as above that $H^{-1}(F) = \partial A \times [0, 1]$. But then $\tilde{f}_2$ is homotopic rel ∂A to a map into $\partial\widetilde{M}_1$. This contradicts our construction of $\tilde{f}_2$. Thus h is essential and the proof of Lemma 5.10 is completed by an application of Theorem 2 above.

LEMMA 5.11. *Let F be a good surface and f: $T \longrightarrow M$ an essential map. Let N be a regular neighborhood of F. Suppose*

(1) $f^{-1}(F)$ is the union of a nonempty collection of simple loops.

(2) f is not homotopic to a map $\bar{f}$: $T \longrightarrow M$ such that $\bar{f}^{-1}(F)$ is the union of a collection of simple loops and $\bar{f}^{-1}(F)$ contains fewer components that $f^{-1}(F)$.

(3) The f image of each component of $T - f^{-1}(F)$ meets both components of $N - F$.

Then there is either an essential map f_1: $K \longrightarrow M$ where K is either a torus or Klein bottle such that the restriction of f_1 to each component of $f_1^{-1}(F)$ and *each component of* $K - f_1^{-1}(F)$ *is a homeomorphism or an essential embedding* $g: (A, \partial A) \longrightarrow (M, \partial M)$. *If the former is the case,* $f_1^{-1}(F)$ *is not empty.*

PROOF. As a consequence of Lemma 5.10, we may assume that F is not the planar surface with three boundary components. It follows from Lemma 5.4 that we may assume that for each loop λ in $f^{-1}(F)$, $f(\lambda)$ is not freely homotopic to a loop in ∂F.

Let $M|F$ be the 3-manifold obtained by splitting M along F. Let $P: M|F \longrightarrow M$ be the natural projection map. Let F^1 and F^2 be the components of $P^{-1}(F)$. Let M_i be homeomorphic to $M|F$ and F_i^1 and F_i^2 be the copies of F^1 and F^2 respectively in ∂M_i for i an integer. Let $(\widetilde{M}, p)$ be the infinite cyclic covering space of M obtained from $\bigcup_{i=-\infty}^{\infty} M_i$ by identifying F_i^1 and F_{i+1}^2 for i an integer. We denote the image of F_i^1 under this identification by F_i are are careful to define the identification so that the natural projection map from $\widetilde{M} - \bigcup_{i=-\infty}^{\infty} F_i$ to M can be extended to the covering map $p: \widetilde{M} \longrightarrow M$. Let $(\widetilde{T}, q)$ be the infinite cyclic covering of T associated with some simple essential loop in $f^{-1}(F)$. Then there is a map $\widetilde{f}: \widetilde{T} \longrightarrow \widetilde{M}$ such that $p\widetilde{f} = fq$.

Let N_i be the submanifold of $\widetilde{M}$ bounded by F_0 and F_i for each positive integer i. Then $\widetilde{f}^{-1}(N_i)$ is an annulus A_i such that $\widetilde{f}(\partial A_i)$ meets both F_0 and F_i. It is a consequence of Lemma 5.8 that an annulus embedded in N_i constructed via a tower argument from the map $\widetilde{f}|A_i$ can meet F_j in only finitely many ways up to free homotopy since $\widetilde{f}(A_i) \cap F_j \subset p^{-1}(f(T) \cap F)$ where $0 \leqslant j \leqslant i$. Of course the number n_0 determined by $f(T) \cap F$ is independent of both i and j above. Let $n = 2n_0 + 2$. Let $f_n = \widetilde{f}|A_n: A_n \longrightarrow N_n$.

It is a consequence of Theorem 3 that we can find an embedding $g_n: (A_n, \partial A_n) \longrightarrow (N_n, \partial N_n)$ such that one component of $g_n(\partial A_n)$ is not freely homotopic in $F_0 \cup F_n$ to a loop in $\partial(F_0 \cup F_n)$ since F is not the planar surface with three boundary components. Suppose a loop λ in $g_n^{-1}(\bigcup_{i=1}^{n-1} F_i)$ is inessential on A_n. Then $g_n(\lambda)$ bounds a disk in $\bigcup_{i=1}^{n-1} F_i$ and it follows from standard arguments that g_n can be modified to obtain a map $g_n^1: (A_n, \partial A_n) \longrightarrow (N_n, \partial N_n)$ which agrees with g_n on all essential loops in $g_n^{-1}(\bigcup_{i=1}^{n-1} F_i)$ so that $g_n^1(A_n) \cap \bigcup_{i=0}^{n} F_i$ contains fewer loops than does $g_n(A_n) \cap \bigcup_{i=0}^{n} F_i$. It follows that we may suppose that all loops in $g_n^{-1}(\bigcup_{i=0}^{n} F_i)$ are essential in A_n. After a general position argument we may suppose that $g_n(A_n)$ crosses F_i at each loop in $g_n(A_n) \cap F_i$.

Let λ be a loop in $g_n^{-1}(F_i)$ for $0 < i < n$. Then by Lemma 5.1, $pg_n(\lambda)$ is not freely homotopic in F to a loop in ∂F since F is good and $pg_n(A)$ shows that $pg_n(\lambda)$ is freely homotopic in M to a loop in F which is not freely homotopic in F to a loop in ∂F.

We observe that n has been chosen so large that there are integers k, l, m such that

(1) $0 < k < l < m < n$.

(2) $p(g_n(A_n) \cap F_k) = p(g_n(A_n) \cap F_l) = p(g_n(A_n) \cap F_m)$.

Thus there are distinct loops λ_1, λ_2, and λ_3 on A_n such that $pg_n(\lambda_1) = pg_n(\lambda_2) = pg_n(\lambda_3)$ and $g_n(\lambda_j) \subset F_k \cup F_1 \cup F_m$ for $j = 1, 2, 3$. We may suppose that λ_3 lies between λ_1 and λ_2 on A_n. Let B_j be the annulus on A_n bounded by $\lambda_j \cup \lambda_3$ for $j = 1, 2$ and $B_3 = B_1 \cup B_2$. Let R be a regular neighborhood of F in M. It follows from a simple argument that there exists a $j = 1, 2$, or 3 such that the image under pg_n of every neighborhood in B_j of ∂B_j meets both components of $R - F$. We suppose that this is the case for B_1.

Let $\mu_1, \ldots, \mu_m$ be the components of $g_n^{-1}(\bigcup_{i=1}^{n-1} F_i) \cap B_1$. Let $C_1, \ldots, C_{m-1}$ be the closures of the components of $B_1 - \bigcup_{i=1}^{m} \mu_i$. If $g_n(\partial C_i)$ lies on a single component F_j of $p^{-1}(F)$ and $g_n | C_i$: $(C_i, \partial C_i) \longrightarrow (\widetilde{M}, F_j)$ is homotopic rel ∂C_i to a map into F_j, we can replace g_n by an embedding g_n^1: $B_1 \longrightarrow \widetilde{M}$ such that $g_n | \partial B_1 = g_n^1 | \partial B_1$ and $g_n^1(B_1) \cap \bigcup_{i=0}^{n} F_i$ contains fewer loops than does $g_n(B_1) \cap \bigcup_{i=0}^{n} F_i$. Thus we may suppose that $g_n | C_i$: $(C_i, \partial C_i) \longrightarrow (\widetilde{M}, F_j)$ is not homotopic rel ∂C_i to a map into F_j since the components of $g_n(\partial B_1)$ lie on distinct components of $p^{-1}(F)$.

Let $\bar{f}$: $K \longrightarrow M$ be the map induced by pg_n: $B_1 \longrightarrow M$. By construction the restriction of $\bar{f}$ to each component of $K - \bar{f}^{-1}(F)$ is an embedding. We will show that $\bar{f}$ is essential.

Let λ be any simple loop in $\bar{f}^{-1}(F)$. Then $f(\lambda)$ is an essential simple loop on F that is not freely homotopic to a loop in ∂F by construction. Since F is good, $f(\lambda)$ is not freely homotopic to a loop in ∂M. It follows from Lemma 5.3 and the construction of $\bar{f}$ that $\bar{f}_*$: $\pi_1(K) \longrightarrow \pi_1(M)$ is monic. Thus $\bar{f}$ is essential and the proof of Lemma 5.11 is complete.

LEMMA 5.12. *Let M be a connected, irreducible 3-manifold such that ∂M is incompressible. Let K_1 be a closed connected surface such that $\chi(K_1) = 0$. Let F_1 be a good surface in M that is not the planar surface with three boundary components. Let f: $K_1 \longrightarrow M$ be a map such that*

(1) *f_*: $\pi_1(K_1) \longrightarrow \pi_1(M)$ is monic.*

(2) *$f^{-1}(F_1)$ is the union of a collection of disjoint simple loops.*

(3) *There is a loop $\lambda \subset f^{-1}(F_1)$ such that $f(\lambda)$ is not freely homotopic in F_1 to a loop in ∂F_1.*

(4) *The components of $K_1 - f^{-1}(F_1)$ are open annuli whose closures we denote by $A_1, \ldots, A_n$.*

(5) *$f(A_1)$ meets only one component of $R - F_1$ (where R is a regular neighborhood of F in M).*

(6) *$f | A_i$ is not homotopic* rel *∂A_i to a map into F_1 for $i = 1, \ldots, n$.*

(7) *f is not homotopic to a map f_1 such that $f_1^{-1}(F_1)$ contains fewer loops than does $f^{-1}(F_1)$.*

Then there is a closed connected surface K such that $\chi(K) = 0$ *and an essential embedding* $\bar{f}: K \longrightarrow M$ *such that* $\bar{f}^{-1}(F_1)$ *contains an essential simple loop* λ *so that* $\bar{f}(\lambda)$ *is not freely homotopic in* F_1 *to a loop in* ∂F_1.

PROOF. Let $g_i: (A, \partial A) \longrightarrow (M, F_1)$ for $i = 1, \ldots, m$ be a maximal collection of pairwise disjoint F_1-essential embeddings such that

(1) $\bigcup_{i=1}^m g_i(A)$ meets only one component of $R - F_1$.

(2) Neither $g_i(c_1)$ nor $g_i(c_2)$ is freely homotopic in F_1 to a loop in ∂M_1 for $i = 1, \ldots, m$.

(3) $g_i(A)$ and $g_j(A)$ are not parallel rel F_1 for $1 \leqslant i < j \leqslant m$.

The existence of this collection, in particular its finiteness is guaranteed by the theorem on p. 60 in [15]. Let R_1 be a regular neighborhood of $\bigcup_{i=1}^m g_i(A) \cup F_1$. Let M_1 and M_2 be the closures of the components of $R_1 - F_1$. We may suppose that M_2 is homeomorphic to $F_1 \times [0, 1]$. Since F_1 does not separate M and $[\mathrm{cl}(\partial M_1 - F_1)]$ is homologous to $[F_1]$, there is a component of $\partial M_1 \cap \mathrm{int}(M)$ whose closure fails to separate M. We denote the closure of this component by F_2. It can be seen that $\chi(F_2) \geqslant \chi(F_1)$. It is a consequence of repeated applications of Lemma 5.2 or of the proof of Lemma 5.2 that F_2 is good. Thus $\chi(F_2) = \chi(F_1)$. Now if T_0 is the closure of a component of $\partial M_1 - (\partial M \cup F_1 \cup F_2)$ it can be seen that $\chi(T_0) = 0$ since $\chi(\partial M_1) = 2{*}\chi(F_1)$ and $T_0 \neq S^2$.

Let R_2 be a regular neighborhood of $\bigcup_{i=1}^m g_i(A)$ in M_1. Let α_i be a spanning arc of A_i for $i = 1, \ldots, n$. Since $f|A_i$ is not homotopic rel ∂A_i to a map into F_1 for $i = 1, \ldots, n$, $f(\alpha_i)$ is not homotopic rel its endpoints to an arc in F_1 for $i = 1, \ldots, n$ by Proposition 3.2.

Suppose that $\bigcup_{i=1}^m g_i(A) \cup f(A_j)$ meets only one component of $N - F_1$. We claim that f is homotopic to a map f_1, under a homotopy constant outside of a neighborhood of A_j, such that

(1) $f_1^{-1}(F_1) = f^{-1}(F_1)$.

(2) $f_1^{-1}(\bigcup_{i=1}^m g_i(\partial A)) \cap A_j$ is empty.

We need only show that there is a homotopy $h_t: (A_j, \partial A_j) \longrightarrow (M, F_1)$ for $t \in [0, 1]$ such that

(1) $h_0 = f|A_j$.

(2) $h_1(\partial A_j) \cap \bigcup_{i=1}^m g_i(\partial A)$ is empty.

After a general position argument, we may suppose that $h_0^{-1}(\bigcup_{i=1}^m g_i(\partial A))$ is a finite set. We assume also that there is no map $h_1(A_j, \partial A_j) \longrightarrow (M, F_1)$ homotopic to $h_0: (A_j, \partial A_j) \longrightarrow (M, F_1)$ such that $h_1^{-1}(\bigcup_{i=1}^m g_i(\partial A))$ contains fewer points than $h_0^{-1}(\bigcup_{i=1}^m g_i(\partial A))$. After a general position argument, we may assume that $h_0^{-1}(\bigcup_{i=1}^m g_i(A)) = J$ is a collection of disjoint simple arcs and loops properly embedded in A_j. We assume also that if h_1 is homotopic to h_0 rel ∂A_j, $h_1^{-1}(\bigcup_{i=1}^m g_i(A))$ contains no fewer components than J.

Suppose that $\lambda \subset J$ is a nullhomeotopic simple loop on A_j. Since $h_0(\lambda)$ is nullhomotopic in $\bigcup_{i=1}^m g_i(A)$, we can modify h_0 on a neighborhood of the disk bounded by λ to obtain a map h_1, homotopic to h_0 rel ∂A_j, that contradicts our choice of h_0. Suppose there is an arc $\beta_1 \subset J$ having both its endpoints on a single component of ∂A_j. Thus there is an arc $\beta_2 \subset \partial A_j$ such that $\beta_1 \cup \beta_2$ bounds a disk $\mathcal{D}$ on A_j. We observe that $h_0(\beta_1)$ is homotopic rel its endpoints to an arc in $\bigcup_{i=1}^m g_i(\partial A)$ and thus $h_0(\beta_2)$ is homotopic rel its endpoints to an arc in $\bigcup_{i=1}^m g_i(\partial A)$. Thus it can be seen that h_0 is homotopic to a map h_1, as above, such that $h_1^{-1}(\bigcup_{n=1}^m g_i(\partial A))$ contains fewer points than $h_0^{-1}(\bigcup_{i=1}^m g_i(\partial A))$. This contradicts our choice of h_0 and we conclude that either $h_0^{-1}(\bigcup_{i=1}^m g_i(\partial A))$ is empty as was to be shown or J is a collection of disjoint spanning arcs of A_j. In the latter case it is not difficult to show that after a homotopy we may suppose that $h_0(J)$ is a nonempty collection of disjoint simple spanning arcs on some of the $g_i(A)$ for $i = 1, \ldots, m$.

Suppose $h_0(J)$ does not lie on one component of $\bigcup_{i=1}^m g_i(A)$. Let $\mathcal{D}$ be the closure of a component of $A_j - J$. We may suppose that $h_0(\mathcal{D} \cap J)$ does not lie on a single component of $\bigcup_{i=1}^m g_i(A)$. It is a consequence of the proof of the loop theorem [12] that there is an embedding $h\colon \mathcal{D} \longrightarrow M$ such that $h(\mathcal{D}) \cap \bigcup_{i=1}^m g_i(A) = h_0(\mathcal{D}) \cap \bigcup_{i=1}^m g_i(A)$ and $h(\mathcal{D}) \cap (F_1 \cup \bigcup_{i=1}^m g_i(A)) = h(\partial \mathcal{D})$. We suppose that $h(\mathcal{D}) \cap g_r(A)$ and $h(\mathcal{D}) \cap g_s(A)$ are nonempty where $1 \leqslant r < s \leqslant m$. We will show that $g_r(A)$ and $g_s(A)$ are parallel. We suppose that $h(\mathcal{D})$ is in general position with respect to ∂M_1. Then $h(\mathcal{D}) \cap \partial M_1$ may be taken to be a simple loop λ. If λ lies on F_2, λ is homotopic to a point on F_2 since F_2 is incompressible in M. It follows that $h(\mathcal{D})$ could be taken to lie in M_1. This is impossible since the retraction of M_1 to $F_1 \cup \bigcup_{i=1}^m g_i(A)$ would yield a retraction of $h(\mathcal{D})$ to $h(\partial \mathcal{D})$.

Thus λ must be on a toroidal boundary component T_1 of M_1. Note that T_1 is a torus and that $g_r(A) \cup g_s(A)$ together with two annuli on F_1 form a compressible torus. Let $\mathcal{U}$ be a regular neighborhood of $T_1 \cup h(\mathcal{D})$. Then one component of $\partial \mathcal{U}$ is a 2-sphere; and since M is irreducible, it can be seen that T_1 bounds a solid torus in M. Since this solid torus meets ∂M_1 only in T_1, it can be seen that $g_r(A)$ and $g_s(A)$ are parallel rel F_1. Thus we may assume that $h_0(J)$ lies on $g_1(A)$.

Let $\hat{M}_1$ be a regular neighborhood of $F_1 \cup g_1(A)$ in M_1. Let $\hat{F}_2$ be a component of $\mathrm{cl}(\partial M_1 - (\partial M \cup F_1))$ that fails to separate M. By Lemma 5.2, $\hat{F}_2$ is good in M, and thus $\hat{F}_2$ is incompressible in M. Since $\partial \hat{M}_1$ can be obtained by removing two annuli from and adding two annuli to the boundary of a regular neighborhood of F_1 in M_1, $\partial \hat{M}_1$ has at most two components if F_1 has boundary and three components if F_1 is closed. We suppose that h_0 is in general position with respect to $\partial \hat{M}_1$. Then $h_0^{-1}(\hat{F}_2)$ is a nonempty collection of simple loops in the complement of J. Let $\mathcal{D}$ be the closure of a component of $A_j - J$ that meets $h_0^{-1}(\hat{F}_2)$. It can be seen that $\mathcal{D} \cap h_0^{-1}(\hat{F}_2)$ is a single simple loop λ. Since $\hat{F}_2$ is incompressible $h_0(\lambda)$ is nullhomotopic on $\hat{F}_2$. It follows that there is a map h_1 of A_j into M such that $h_1(\mathcal{D}) \subset \hat{F}_2$ and $h_1|(A_j - \mathcal{D}) = h_0|(A_j - \mathcal{D})$. Thus after a homotopy rel ∂A_j, we may suppose that $h_0(\mathcal{D}) \subset \hat{F}_2$ since $\pi_2(M) = 0$.

Let $h = h_0|\mathcal{D}$. Since there is a deformation of $\hat{M}_1$ onto $F_1 \cup g_1(A)$, h is homotopic rel $\partial\mathcal{D}$ ($J \cup \partial A_j$, if J is connected) to a map $\hat{h}$ into $F_1 \cup g_1(A)$. It follows from standard arguments that $\hat{h}$ is homotopic rel $\partial\mathcal{D}$ to a map $\hat{h}_1\colon \mathcal{D} \longrightarrow F_1 \cup g_1(A)$ such that $\hat{h}_1^{-1}(g_1(\partial A))$ is a collection of disjoint simple arcs properly embedded in $\mathcal{D}$.

Suppose β_1 is a simple arc in $\hat{h}_1^{-1}(g_1(\partial A))$ properly embedded in $\mathcal{D}$. Then β_1 cuts off a disk $\mathcal{D}_1$ on $\mathcal{D}$ such that $\mathcal{D}_1 \cap \hat{h}_1^{-1}(g_1(\partial A)) = \beta_1$ for $\partial\mathcal{D} \cap \hat{h}_1^{-1}(g_1(\partial A))$ contains exactly four (or two) points. Let β_2 be the closure of $\partial\mathcal{D}_1 - \beta_1$. Then $\hat{h}_1(\beta_2)$ is homotopic rel its endpoints to $\hat{h}_1(\beta_1)$. But $\beta_2 \subset \partial A_j$ and $h_0(\beta_2)$ is now seen to be homotopic rel its endpoints to an arc on $g_1(\partial A)$. This contradicts our choice of h_0.

Our claim follows and we suppose that $f^{-1}(\bigcup_{i=1}^m g_i(\partial A)) \cap A_j$ is empty for each j such that $\bigcup_{i=1}^m g_i(A) \cup f(A_j)$ meets only one component of $N - F_1$ where $1 \leqslant j \leqslant n$.

Suppose that there is an integer j where $1 \leqslant j \leqslant n$ such that $f(\alpha_j)$ meets both components of $N - F_1$. We claim that there is a map f_1 homotopic to f under a homotopy constant outside of a neighborhood of A_j such that $A_j \cap f_1^{-1}(\bigcup_{i=1}^m g_i(A))$ is a collection (possibly empty) of disjoint simple essential loops and $f_1^{-1}(\bigcup_{i=1}^m g_i(A)) \subseteq f^{-1}(\bigcup_{i=1}^m g_i(A))$.

After a general position argument, we may suppose that $f^{-1}(\bigcup_{i=1}^m g_i(A)) \cap A_i = J$ is the union of a collection of disjoint simple arcs and loops properly embedded in A_j. After a homotopy we also assume that f has been chosen so that if f_1 is homotopic to f as above, the number of points in $f^{-1}(\bigcup_{i=1}^m g_i(\partial A)) \cap A_j$ is no greater than that in $f_1^{-1}(\bigcup_{i=1}^m g_i(\partial A)) \cap A_j$, and the number of components in $f_1^{-1}(\bigcup_{i=1}^m g_i(A)) \cap A_j$ is not less than the number in J.

Suppose that $\lambda \subset J$ is a simple nullhomotopic loop. Then as in the proof above we can modify f to find a map f_1 homotopic to f such that the number of components in $f_1^{-1}(\bigcup_{i=1}^m g_i(A)) \cap A_j$ is less than the number in J. Since this contradicts our choice of f, we suppose that each simple loop in J is essential.

Suppose that β_1 is an arc in J. Since $f(\beta_1)$ lies on one of the $g_i(A)$ for $i = 1, \ldots, m$, $f(\beta_1)$ meets only one component of $N - F_1$. It follows that $\partial\beta_1$ lies on a single component of ∂A_j. Thus β_1 cuts off a disk $\mathcal{D}$ on A_j. Let $\beta_2 = \partial\mathcal{D} \cap \partial A_j$. Since $f(\beta_2)$ is homotopic rel its boundary to an arc in $\bigcup_{i=1}^m g_i(\partial A)$, it follows as above that there is a map f_1 homotopic to f such that $f_1^{-1}(F_1) = f^{-1}(F_1)$ and $f_1^{-1}(\bigcup_{i=1}^m g_i(\partial A)) \cap A_j$ contains fewer points than does

$f^{-1}(\bigcup_{i=1}^m g_i(\partial A) \cap A_j)$. Since this homotopy can be taken to be constant on the complement of a regular neighborhood of A_j, the existence of f_1 contradicts our choice of f. This establishes our claim.

Suppose $\bar{f}$: $(A, \partial A) \longrightarrow (M, F_1)$ is a map such that $\bar{f}(A) \cap \bigcup_{i=1}^m g_i(A)$ is empty and $\bar{f}(A) \cup \bigcup_{i=1}^m g_i(A)$ meets only one component of $R - F_1$. Let F^* be the closure of the complement in F_1 of a regular neighborhood in F_1 of $\bigcup_{i=1}^m g_i(\partial A)$. Suppose that $\bar{f}(c_1)$ is not freely homotopic in F^* to a loop in ∂F^*, the component of F^* on which $\bar{f}(c_1)$ lies is not the planar surface with three boundary components, and $\bar{f}(\alpha)$ is not homotopic rel its endpoints to an arc on F_1. Then we claim the collection g_i for $i = 1, \ldots, m$ is not maximal.

If $\bar{f}(c_1)$ and $\bar{f}(c_2)$ lie on distinct components of F^*, there is an embedding g_{m+1}: $(A, \partial A) \longrightarrow (M, F_1)$ such that $\bigcup_{i=1}^{m+1} g_i(A)$ meets only one component of $R - F_1$, g_{m+1*}: $\pi_1(A) \longrightarrow \pi_1(M)$ is monic, and $g_{m+1}(c_1)$ is not freely homotopic in F^* to a component of ∂F^* by Theorem 3. If $g_{m+1}(\alpha)$ is homotopic rel its endpoints to an arc in F^*, g_{m+1} is homotopic rel ∂A to a map into F_1 by Proposition 3.2 since $\pi_2(M) = 0$ and F_1 is incompressible. But then $g_{m+1}(c_1)$ is freely homotopic in F_1 to $g_{m+1}(c_2)$ and it follows from standard arguments that g_{m+1} is homotopic in F^* to a loop in ∂F^* since each loop in ∂F^* is essential in F_1. Clearly $g_{m+1}(A)$ is not parallel to $g_i(A)$ for $1 \leqslant i \leqslant m$ since the loop $g_{m+1}(c_1)$ is not freely homotopic to a loop in ∂F^*. Thus g_{m+1} extends the collection $\{g_i\colon 1 \leqslant i \leqslant m\}$ and we suppose that $\bar{f}(c_1)$ and $\bar{f}(c_2)$ lie on a single component of F^*.

We may suppose that $\bar{f}^{-1}(\partial M_1 - F_1)$ is the union of two essential simple loops since $\bar{f}(A) \cap \bigcup_{i=1}^m g_i(A)$ is empty. Since the component of F^* on which $\bar{f}(c_1 \cup c_2)$ lies is not an annulus, $\bar{f}(\bar{f}^{-1}(\partial M_1 - F_1))$ lies on F_2. Now $\bar{f}$ induces a map $\bar{f}_1$: $(A, \partial A) \longrightarrow (M, F_2)$ by restriction of $\bar{f}$ to the closure of the central component of $A - \bar{f}^{-1}(F_2)$. Suppose $\bar{f}_1(\alpha)$ is homotopic rel its endpoints to an arc on F_2. Since F_2 is incompressible and $\pi_2(M) = 0$, $\bar{f}_1$ and $\bar{f}$ are homotopic rel ∂A to a map into M_1. Since there is a deformation retraction of M_1 to $F_1 \cup \bigcup_{i=1}^m g_i(A)$, $\bar{f}$ is homotopic rel ∂A to a map into $F_1 \cup \bigcup_{i=1}^m g_i(A)$ and $\bar{f}(c_1)$ is homotopic in F^* to a loop in ∂F^*.

Thus we may assume $\bar{f}_1(\alpha_1)$ is not homotopic rel its endpoints to a map into F_2. It is now a consequence of Theorem 2 that there is an F_2-essential embedding g: $(A, \partial A) \longrightarrow (M, F_2)$ such that

(1) $g(\partial A)$ lies in a regular neighborhood in F_2 of $\bar{f}_1(\partial A)$.

(2) $g^{-1}(M_1) = \partial A$.

We wish to replace g by a map g_{m+1}: $(A, \partial A) \longrightarrow (M, F_1)$ that extends the collection of g_i for $i = 1, \ldots, m$. We observe that there are disjoint simple loops λ_1 and λ_2 on the component of F^* on which $\bar{f}(c_1 \cup c_2)$ lies such that $\lambda_j \cup g(c_j)$ bounds an embedded annulus for $j = 1, 2$ because of the local product structure between F^* and portions of ∂M_1. After the usual argument we may assume that these annuli are disjoint. Let $g_{m+1}\colon (A, \partial A) \longrightarrow (M, F_1)$ be an embedding such that $g_{m+1}(A)$ is the union of $g(A)$ and the two annuli mentioned above.

Clearly $g_{m+1*}\colon \pi_1(A) \longrightarrow \pi_1(M)$ is monic. If $g_{m+1}(\alpha)$ is homotopic rel its endpoints to an arc on F_1, g_{m+1} is homotopic rel ∂A to a map into F_1 by Proposition 3.2 since F_1 is incompressible and $\pi_2(M) = 0$. Suppose that $H\colon A \times [0, 1] \longrightarrow M$ is such a homotopy. Using standard techniques we may assume $H^{-1}(F_2)$ is an incompressible surface properly embedded in $A \times [0, 1]$. Since $H^{-1}(F_2) \cap \partial(A \times [0, 1])$ is a pair of disjoint essential loops, $H^{-1}(F_2)$ is an embedded annulus. By Proposition 3.1 in [1] this annulus is parallel to an annulus in $\partial(A \times [0, 1])$ and it can be seen that g must have been homotopic rel ∂A to a map into F_2. Thus we suppose that $g_{m+1}(\alpha)$ is not homotopic rel its endpoints to an arc in F_1.

If $g_{m+1}(A)$ is parallel rel F_1 to $g_j(A)$ where $1 \leqslant j \leqslant m$, there is an embedding $H\colon A \times [0, 1] \longrightarrow M$ such that

(1) $H(A \times \{0\}) = g_{m+1}(A)$.

(2) $H(A \times \{1\}) = g_j(A)$ for some j where $1 \leqslant j \leqslant m$.

(3) $H(\partial A \times [0, 1]) \subset F_1$.

But then $H^{-1}(F_2)$ is an annulus embedded in $A \times [0, 1]$, and it can be seen as above that $g(A)$ is homotopic to a map into F_2. It follows that $g_{m+1}\colon (A, \partial A) \longrightarrow (M, F_1)$ extends the collection of g_i for $i = 1, \ldots, m$, and the proof of our claim is complete.

Let F_3 be the closure of $\operatorname{int}(M) \cap \partial M_2 - F_1$. Let $g_i'\colon (A, \partial A) \longrightarrow (M, F_3)$ for $i = 1, \ldots, m'$ be a maximal collection of disjoint F_3-essential embeddings such that

(1) $g_i'(A) \cap (M_2 - F_3)$ is empty for $i = 1, \ldots, m'$.

(2) $g_i'(A)$ and $g_j'(\alpha)$ are not parallel rel F_3 for $1 \leqslant i < j \leqslant m'$.

After an argument similar to the one given above we may assume

(1) $f^{-1}(F_1 \cup F_3)$ is a collection of simple essential loops,

(2) $f(K_1) \cap (\bigcup_{i=1}^m g_i(\partial A) \cup \bigcup_{i=1}^{m'} g_i'(\partial A))$ is empty.

(3) $f^{-1}(\bigcup_{i=1}^m g_i(A))$ and $f^{-1} \bigcup_{i=1}^{m'} g_i'(A)$ are unions of nonempty collections of disjoint simple essential loops.

(4) There is no map f_1 homotopic to f such that $f_1^{-1}(F_1 \cup F_3)$ contains fewer loops than does $f^{-1}(F_1 \cup F_3)$.

We now let $A_1, \ldots, A_n$ be the closures of the components of $K_1 - f^{-1}(F_1 \cup F_3)$. Let $\overline{F}_1$ $(\overline{F}_3)$ be the closure of the complement of a regular neighborhood of $\bigcup_{i=1}^m g_i(\partial A)$ $(\bigcup_{i=1}^{m'} g_i'(\partial A))$ in F_1 (F_3). We may suppose that $f(K_1) \cap F_1 \subset \overline{F}_1$ and $f(K_1) \cap F_3 \subset \overline{F}_3$. If $f(A_j) \subset M_2$, we may suppose that f carries one component of ∂A_j to $\overline{F}_1$ and the other to $\overline{F}_3$ for $1 \leqslant j \leqslant n$. Let $\lambda_1, \ldots, \lambda_n$ be the components of $f^{-1}(F_1 \cup F_3)$. We may suppose that $A_i \cap$

$A_{i+1} = \lambda_i$ for $i = 1, \ldots, n-1$ and $A_n \cap A_1 = \lambda_n$. Now if λ_{i-1} and λ_i are the components of ∂A_i and $f(\lambda_{i-1} \cup \lambda_i)$ lies on $\bar{F}_1$, it follows from the claim above that $f(\lambda_{i-1})$ $(f(\lambda_i))$ is freely homotopic in $\bar{F}_1$ to a loop in $\partial \bar{F}_1$ or the component of $\bar{F}_1$ on which $f(\lambda_{i-1})$ $(f(\lambda_i))$ lies is planar with three boundary components. The statement above is also true if we replace $\bar{F}_1$ by $\bar{F}_3$.

Let K be a closed surface of genus one of unspecified orientability. We claim there is an essential map $\bar{f}$: $K \longrightarrow M$ such that $\bar{f}^{-1}(F_1 \cup F_3)$ is a nonempty collection of simple essential loops, $\bar{f} | \bar{f}^{-1}((F_1 - \bar{F}_1) \cup (F_3 - \bar{F}_3))$ is a homeomorphism, and the restriction of $\bar{f}$ to each component of $K - \bar{f}^{-1}(F_1 \cup F_3)$ is a homeomorphism. We will construct this map by finding a sequence of $(F_1 \cup F_3)$-essential embeddings $\bar{f}_i$: $(A, \partial A) \longrightarrow (M, (F_1 - \bar{F}_1) \cup (F_3 - \bar{F}_3))$ for $1 \leqslant i$ such that

(1) $\bar{f}_i(c_2) = \bar{f}_{i+1}(c_1)$ for $1 \leqslant i$.

(2) $\bar{f}_i(c_1)$ is not freely homotopic to a loop in ∂M for $0 < i$ and fitting together a finite sequence of these maps to form $\bar{f}$.

We may suppose that $f(\lambda_1 \cup \lambda_n) \subset \bar{F}_1$ and that for $1 \leqslant j \leqslant n$ the image under f of λ_j is a simple loop whenever $f(\lambda_j)$ is freely homotopic in $\bar{F}_1 \cup \bar{F}_3$ to a loop in $\partial(\bar{F}_1 \cup \bar{F}_3)$. Then there is an embedding $\bar{f}_1$: $(A, \partial A) \longrightarrow (M, F_1)$ such that $\bar{f}_1^{-1}(M_2) = \partial A$, $\bar{f}_1(\alpha)$ is not homotopic rel its endpoints to an arc on F_1, and $\bar{f}_{1*}$: $\pi_1(A) \longrightarrow \pi_1(M)$ is monic by Theorem 2 or Lemma 5.9. If $f(\lambda_1)$ $(f(\lambda_n))$ is not freely homotopic in $\bar{F}_1$ to a loop in $\partial \bar{F}_1$, the component of $\bar{F}_1$ on which $f(\lambda_1)$ $(f(\lambda_n))$ lies is a planar surface with three boundary components by the claim above which allows us to extend the collection $\{g_i$: $i = 1, \ldots, m\}$; so we may suppose that $\bar{f}_1(c_1 \cup c_2)$ lies in $F_1 - \bar{F}_1$. Since F_1 is not the planar surface with three boundary components and no loop essential in $\bar{F}_1$ is inessential in F_1, we may suppose that $\bar{f}(c_1)$ is not freely homotopic to a loop in ∂F_1. Thus since F_1 is good, $\bar{f}_1(c_1)$ is not freely homotopic to a loop in ∂M. The reader should notice the trick that allows us to assume that $\bar{f}_1(c_1 \cup c_2) \subset F_1 - \bar{F}_1$ as it will be used again below without further justification. We shall say that $\bar{f}_1$ *is constructed using* $f | A_1$. Since a regular neighborhood of λ_i for $i = 1, \ldots, n$ meets both $f^{-1}(\mathrm{int}(M_2))$ and $f^{-1}(M - M_2)$, there is a smallest integer j such that $f(\lambda_j) \cup f(\lambda_{j+1})$ lies on $\bar{F}_3$. Let $\bar{A} = \bigcup_{i=2}^{j} A_i$ and $g = f | \bar{A}$. Let $(\widetilde{M}, p)$ be the infinite cyclic covering of M associated with F_1. Let $\widetilde{g}$: $\bar{A} \longrightarrow \widetilde{M}$ be a map such that $p\widetilde{g} = g$. Then it can be seen that if $\widetilde{F}$ is a component of $p^{-1}(F_1 \cup F_3)$, $\widetilde{g}^{-1}(\widetilde{F})$ contains at most a single simple loop. Let $\widetilde{F}_1$ and $\widetilde{F}_3$ respectively be the components of $p^{-1}(F_1 \cup F_3)$ on which $\widetilde{g}(\lambda_1)$ and $\widetilde{g}(\lambda_j)$ lie. Let $\widetilde{M}_1$ be the closure of the component of $\widetilde{M} - (\widetilde{F}_1 \cup \widetilde{F}_3)$ in which $\widetilde{g}(\mathrm{int}(\bar{A}))$ lies.

It is a consequence of Theorem 3 or the theorem in [14] that there is an embedding $\hat{g}$: $\bar{A} \longrightarrow \widetilde{M}_1$ such that

(1) $\hat{g}_*$: $\pi_1(\bar{A}) \longrightarrow \pi_1(\widetilde{M}_1)$ is monic.

(2) $p\hat{g}(\lambda_1) = \bar{f}_1(c_2)$.

(3) $p\hat{g}(\lambda_j) \subset F_3 - \bar{F}_3$.

(4) $\hat{g}^{-1}p^{-1}(F_1 \cup F_3)$ is a collection of essential simple loops.

(5) $p\hat{g}(\lambda_i)$ lies in a regular neighborhood in $F_1 \cup F_3$ of $f(\lambda_i)$ for $i = 2, \ldots, j-1$.

It is a consequence of (5) above that we may suppose $p\hat{g}(A) \cap ((F_1 - \bar{F}_1) \cup (F_3 - \bar{F}_3)) = p\hat{g}(\partial\bar{A})$. We next apply the theorem [14] to the map $p\hat{g}$ to find an embedding $h\colon A \longrightarrow M$ such that $h(\partial A) = p\hat{g}(\partial\bar{A})$. After the usual argument we may suppose that

(1) $h^{-1}(F_1 \cup F_3)$ is a collection of simple essential loops.

(2) $h^{-1}((F_1 - \bar{F}_1) \cup (F_3 - \bar{F}_3)) = \partial A$.

(3) There does not exist a map h_1 homotopic to h rel ∂A such that $h_1^{-1}(F_1 \cup F_3)$ contains fewer loops than $h^{-1}(F_1 \cup F_3)$.

Let $\bar{f}_2 = h$.

We next apply Lemma 5.9 or Theorem 2 to $f|A_{j+1}$ to find an embedding $\bar{f}_3'\colon (A, \partial A) \longrightarrow (M, F_3)$ such that

(1) $\bar{f}_3'(c_1) = \bar{f}_2(c_2)$.

(2) $\bar{f}_3'(\alpha)$ is not homotopic rel its endpoints to an arc in F_3.

(3) $\bar{f}_3'(c_2)$ lies in a regular neighborhood in F_3 of $f(\lambda_{j+1})$.

Since $\bar{f}_3'(c_2)$ is simple, we may replace $\bar{f}_3'$ by an embedding $\bar{f}_3$ satisfying (1) and (2) above such that $\bar{f}_3(c_2)$ lies on $F_3 - \bar{F}_3$.

Suppose we have constructed $\bar{f}_i$ and $\bar{f}_{i+1}$. We then use the same technique to construct $\bar{f}_{i+2}$ as that used to construct $\bar{f}_i$ except that after we use $f|A_n$ to construct some $\bar{f}_k$, we will need to use $f|A_1$ to construct $\bar{f}_{k+1}$. Similarly after we reuse $f|A_1$ to construct some $\bar{f}_k$, we must reuse $f|\bar{A}$ to construct $\bar{f}_{k+1}$, etc. Note that $\bar{f}_i(c_1)$ and $\bar{f}_i(c_2)$ are simple loops freely homotopic in the closure of $(F_1 - \bar{F}_1 \cup F_3 - \bar{F}_3)$ to a component of $\partial(\bar{F}_1 \cup \bar{F}_3)$. We suppose $\bar{f}_1, \bar{f}_{1+k}, \bar{f}_{1+2k}$, and $\bar{f}_{1+3k}$ are constructed using $f|A_1$. Since $f(\lambda_n)$ is freely homotopic in $\bar{F}_1$ to a loop on $\partial\bar{F}_1$ or the component of $\bar{F}_1$ on which $f(\lambda_n)$ lies is a planar surface with three boundary components, some pair of loops among $\bar{f}_1(c_1)$, $\bar{f}_{1+k}(c_1), \bar{f}_{1+2k}(c_1)$, and $\bar{f}_{1+3k}(c_1)$ must be freely homotopic. It will be seen below that which pair is freely homotopic is not relevant so we may suppose for our convenience that $\bar{f}_1(c_1)$ and $\bar{f}_{k+1}(c_1)$ are freely homotopic and, after a homotopy, $\bar{f}_k(c_2) = \bar{f}_1(c_1)$. We suppose that K is the union of annuli $\bar{A}_1, \ldots, \bar{A}_k$ such that $\bar{A}_i \cap \bar{A}_{i+1} = \bar{\lambda}_i$ is a component of $\partial\bar{A}_i$ for $i = 1, \ldots, k-1$ and $\bar{A}_k \cap \bar{A}_1 = \bar{\lambda}_k$. We define $\bar{f}\colon K \longrightarrow M$ by $\bar{f}(\bar{A}_i) = \bar{f}_i(A)$ for $1 \leqslant i \leqslant k$ so that $\bar{f}$ will be continuous and note that any collection of embeddings as the $\bar{f}_i$ determine a singular map as $\bar{f}$. After a small motion of $\bar{f}$, we may suppose that $\bar{f}|\bar{f}^{-1}((F_1 - \bar{F}_1) \cup (F_3 - \bar{F}_3))$ is a homeomorphism.

We wish to show that $\bar{f}$ is essential. We observe that $\bar{f}(\bar{\lambda}_j)$ is freely homotopic to $\bar{f}_i(c_1)$ for $1 \leqslant i$ and $1 \leqslant j \leqslant k$ so the loop $\bar{f}(\bar{\lambda}_j)$ is not freely homotopic to a loop on ∂M since F_1 is good. It is a consequence of Lemma 5.4 that $\bar{f}_*$: $\pi_1(K) \longrightarrow \pi_1(M)$ is monic since $\bar{f}_{1*}$: $\pi_1(A) \longrightarrow \pi_1(M)$ is monic. Thus $\bar{f}$ is essential and our claim is established.

We complete the proof of Lemma 5.12 by applying Lemma 5.13 below.

LEMMA 5.13. *Let F be a good surface in M, K a closed, connected surface of genus one, and f: $K \longrightarrow M$ an essential map such that*

(1) *$f^{-1}(F)$ is the union of a collection of disjoint simple essential loops $\lambda_1, \lambda_2, \ldots, \lambda_n$ where $n \geqslant 2$.*

(2) *$f(\lambda_1) \cap \bigcup_{i=2}^n f(\lambda_i)$ is empty.*

(3) *$f|\lambda_i$ is a homeomorphism for $i = 1, \ldots, n$ and $f(\lambda_1)$ is not freely homotopic on F to a loop in ∂F.*

(4) *The closures of the components of $K - \bigcup_{i=1}^n \lambda_i$ are annuli $A_1, \ldots, A_n$.*

(5) *$f|\text{int}(A_i)$ is a homeomorphism.*

(6) *f: $(A_i, \partial A_i) \longrightarrow (M, F)$ is an F-essential map.*

Then there is a closed connected surface K_1 of genus one and an essential embedding g: $K_1 \longrightarrow M$ such that $g^{-1}(F)$ is a nonempty collection of essential loops $\lambda_1^*, \ldots, \lambda_m^*$ and $g(\lambda_1^*)$ is not freely homotopic on F to a loop in ∂F.

PROOF. Let $X(f) = \bigcup_{i\neq j; 1 \leqslant i<j\leqslant n}(f(\lambda_i) \cap f(\lambda_j))$. We assume that f has been chosen so that

(a) f satisfies conditions (1)–(6) above.

(b) $f(\lambda_i) \cap f(\lambda_j) \cap f(\lambda_k)$ is empty if $1 \leqslant i < j < k \leqslant n$.

(c) if f_1 is homotopic to f and satisfies conditions (a) and (b) above, the cardinality of $X(f)$ is less than or equal to that of $X(f_1)$.

It follows from standard arguments that $X(f)$ is a finite set. We claim $X(f)$ is empty.

After a homotopy that keeps $f^{-1}(F)$ fixed we may suppose that $\text{cl}(f^{-1}f(A_j) \cap \text{int } A_i)$ is the union of collection of disjoint simple arcs and loops properly embedded in A_i where $1 \leqslant i, j \leqslant n$ and $i \neq j$. Suppose that $X(f)$ is not empty. Since $f(\lambda_1) \cap X(f)$ is empty there are integers i and j where $i \neq j$ such that $\text{cl}(f^{-1}f(A_i) \cap \text{int}(A_j))$ contains an arc β_1 having both its endpoints on a single component of ∂A_j. Now β_1 cuts off a disk $\mathcal{D}_1$ on A_j. Note that $f|\mathcal{D}_1$ is an embedding. After the usual arguments, we may suppose that $\mathcal{D}_1 \cap f^{-1}f\,\text{int}(A_i) = \text{int}(\beta_1)$. Let β_2 be the arc on A_i such that $f(\beta_1) = f(\beta_2)$. Now β_2 cuts off a disk $\mathcal{D}_2$ on A_i and $f(\mathcal{D}_1) \cup f(\mathcal{D}_2)$ is a disk embedded in M. Let $\gamma_1 = \text{cl}(\partial\mathcal{D}_1 - \beta_1)$ and $\gamma_2 = \text{cl}(\partial\mathcal{D}_2 - \beta_2)$. Since F is incompressible in M, $f(\gamma_1) \cup f(\gamma_2)$ bounds a disk $\mathcal{D}$ on F. After the usual argument, we may suppose that each arc in $f(\lambda_i)$ for $i = 1, \ldots, n$ properly embedded in $\mathcal{D}$ meets both $f(\gamma_1)$ and $f(\gamma_2)$. But it can now be seen that f was not chosen so that $X(f)$ contains a minimal number of points since there is a homotopy f_t: $K \longrightarrow M$ such that $f_0 = f$, $f_t^{-1}(F) = \bigcup_{i=1}^n \lambda_i$ for $t \in [0, 1]$, and the cardinality of $X(f_1)$ is less than that of $X(f)$. This establishes our claim.

We continue the proof of Lemma 5.13 with the added assumption that $f|f^{-1}(F)$ is a homeomorphism. We suppose that f has been chosen subject to the further restraint that $f^{-1}(F)$ contains as few essential loops as possible but at least one loop. Let R be a regular neighborhood of F and F_1 and F_2 the closures of the components of bd(R) − bd(M). We may suppose that $f|f^{-1}(R)$ is a homeomorphism. Let $B_1, \ldots, B_n$ be the closures of the components of $K - f^{-1}(R)$. Note that $f|B_i$ is a homeomorphism for $i = 1, \ldots, n$.

Suppose that $i \neq j$, $f(\partial B_i) \cup f(\partial B_j)$ bounds disjoint annuli $\bar{A}_1$ and $\bar{A}_2$ in $F_1 \cup F_2$, and $f(\partial B_i)$ meets both $\partial \bar{A}_1$ and $\partial \bar{A}_2$. Then we can find $(F_1 \cup F_3)$-essential embeddings g_1, g_2: $(A, \partial A) \longrightarrow (\text{cl}(M - R), f(\partial B_i \cup \partial B_j))$ that are parallel rel $(F_1 \cup F_3)$. Note that it is a consequence of Lemma 5.3 that $f(K - (B_i \cup B_j)) \cup g_1(A) \cup g_2(A)$ determines one or two essential maps of a closed connected surface of genus one. Thus we may suppose that if B_i and B_j are as above $f(B_i)$ is parallel to $f(B_j)$ rel $(F_1 \cup F_3)$.

Suppose that $1 \leqslant i < j < k \leqslant n$ and $f(\lambda_i)$, $f(\lambda_j)$ and $f(\lambda_k)$ lie on a single annulus embedded in F. Let f_1 be a map homotopic to f such that $f_1(\lambda_i) = f_1(\lambda_j) = f_1(\lambda_k)$, $f_1^{-1}(F) = f^{-1}(F)$, and $f_1|A_v$ is a homeomorphism for $v = 1, \ldots, n$. Let A_1^*, A_2^*, and A_3^* be the closures of the components of $K - (\lambda_i \cup \lambda_j \cup \lambda_k)$. Then it can be seen that $f_1|A_1^*$ and $f_1|A_2^*$ or $f_1|(A_1^* \cup A_2^*)$ determines an essential map g: $K_1 \longrightarrow M$ such that $g^{-1}(F)$ contains fewer loops than does $f^{-1}(F)$; note that such a map will be determined by $f_1|A_1^*$ if f_1 does not carry a regular neighborhood of ∂A_1^* in A_1^* into a single component of $R - F$.

It follows that we may suppose that not more than two of the λ_i are mapped by f into a single annulus on F.

Suppose that $f(B_i)$ and $f(B_j)$ are in general position with respect to one another where $i \neq j$ and that $f(B_i) \cap f(B_j)$ contains an essential loop. Let A_1^* and A_2^* be the closures of the components of $K - (B_i \cup B_j)$. Suppose $f(\partial A_1^*)$ meets both F_1 and F_2. It can be seen that there is an annulus B_1^* properly embedded in $M - \text{int}(R)$ such that $\partial B_1^* = f(\partial A_1^*)$. But B_1^* must be $(F_1 \cup F_3)$-essential and it is a consequence of Lemma 5.3 that we can construct an essential map f_1: $K_1 \longrightarrow M$ such that $f_1(K_1) = B_1^* \cup f(A_1^*)$ and $f_1^{-1}(F)$ contains fewer loops than $f^{-1}(F)$. This contradicts our construction of f.

Thus $f(\partial A_1^*)$ may be supposed to lie on F_1. If the loops in $f(\partial A_1^*)$ do not bound an annulus on F_1, we can find an annulus B_1^* properly embedded in $M - \text{int}(R)$ such that $\partial B_1^* = f(\partial A_1^*)$. Note that B_1^* is F_1-essential since otherwise it is a consequence of Proposition 3.2 that ∂B_1^* bounds an annulus on F_1. Now there is an essential map f_1: $K_1 \longrightarrow M$ such that $f_1(K_1) = B_1^* \cup f(A_1^*)$ and $f_1^{-1}(F)$

contains fewer loops than $f^{-1}(F)$. Since this contradicts our construction of F, we may suppose that $f(\partial A_1^*)$ bounds an annulus on F_1. Similarly $f(\partial A_2^*)$ bounds an annulus on $F_1 \cup F_2$. It follows that $f(\partial B_1) \cup f(\partial B_2)$ bounds a pair of disjoint annuli on $F_1 \cup F_2$ and that $f(B_1)$ meets both of these annuli. But then in our construction of f, $f(B_1)$ and $f(B_2)$ were chosen to be parallel rel $(F_1 \cup F_2)$. This is a contradiction and $f(B_i) \cap f(B_j)$ contains no essential loops if $i \neq j$.

Since $\pi_2(M) = 0$, it can be shown that there is a map f_1 homotopic to f rel R such that $f_1(B_j) \cap f_1(B_i)$ is empty if $1 \leqslant i < j \leqslant n$. Lemma 5.13 follows.

VI. **The torus theorem.** In this section we state and prove Waldhausen's "torus theorem."

THEOREM 4. *Let M be a compact, irreducible 3-manifold with nonvacuous boundary. If M admits an essential map of a torus, M admits an essential embedding of either a torus or an annulus.*

PROOF. Let $f\colon T \longrightarrow M$ be an essential map. Suppose $\mathcal{D}$ is a disk properly embedded in M. We suppose that f is in general position with respect to $\mathcal{D}$. Then $f^{-1}(\mathcal{D})$ can contain no essential simple loops since $f_*\colon \pi_1(T) \longrightarrow \pi_1(M)$ is monic. It is not difficult to see that f is homotopic to a map $f_1\colon T \longrightarrow M$ such that $f_1(T) \cap \mathcal{D}$ is empty since $\pi_2(M) = 0$. It follows that we may assume that ∂M is incompressible.

Let $M = M_1$ and M_i, $F_i \subset M_i$, $\mathcal{U}(F_i) \subset M_i$ for $1 \leqslant i \leqslant n$ be a special hierarchy for M_1 with respect to f. We may suppose that k is an integer, $f(T) \subset M_k$, and if f_1 is any map homotopic to f such that $f_1(T) \subset M_k$, $f_1(T) \cap F_k$ is not empty. We claim that either M_k admits an essential map or f is homotopic to a map into ∂M_k.

Since f may be chosen so that $f(T)$ does not meet any disk $\mathcal{D}$ properly embedded in M_k, we may suppose that F_k is not a disk and ∂M_k is incompressible in M_k. Suppose that $f\colon T \longrightarrow M_k$ is not essential. Let λ_1 and λ_2 be simple loops on T which meet at a single point and cross at that point. Then f is homotopic in M_k to a map f_1 such that $f_1(\lambda_1) \subset \partial M_k$. We split T along λ_1 to obtain an annulus A. Now f_1 induces a map $\bar{f}\colon (A, \partial A) \longrightarrow (M_k, \partial M_k)$. If $\bar{f}$ is essential, our claim is established. Otherwise by Proposition 3.2, $\bar{f}$ is homotopic rel ∂A to a map into ∂M_k since ∂M_k is incompressible and $\pi_2(M_k) = 0$. It follows that f is homotopic to a map into ∂M_k and our claim is established.

If $f_1\colon T \longrightarrow \partial M_k$ is a map homotopic to f and $T_1 = f_1(T)$, T_1 is a torus since ∂M_k is incompressible in M_k, $\pi_1(M_k) \longrightarrow \pi_1(M)$ is monic, and $f_{1*}(\pi_1(T)) \cong Z \oplus Z \subseteq \pi_1(T_1)$. Thus it can be seen that T_1 is incompressible in M. Since f_1 is essential, T_1 is essential in M_1 and Theorem 4 follows.

Thus we may suppose that $f\colon T \longrightarrow M_k$ is essential in M_k. It is a consequence of Lemma 5.7 that if M_k admits an essential embedding of an annulus or a torus, $M_1 = M$ admits an essential embedding of an annulus or a torus. Thus we may proceed with the added hypothesis that: if f_1 is any map homotopic to f and F is any good surface in M, $f_1(T) \cap F$ is not empty.

If f is W-essential, Theorem 4 is a consequence of Lemma 5.4. If F is planar and $\chi(F) = -1$, Theorem 4 is a consequence of Lemma 5.10.

Let F_1 be a good surface properly embedded in M. After a general position argument, we may suppose that $f^{-1}(F_1)$ is the union of a collection of disjoint simple loops. We assume that f has been chosen so that the number of loops in this collection is minimal. Suppose some loop $\lambda \subset f^{-1}(F_1)$ is inessential. Let the disk $\mathcal{D}$ be a regular neighborhood of the disk bounded by λ on T. Since F_1 is incompressible, $f(\lambda)$ is inessential on F_1 and $f(\partial \mathcal{D})$ is nullhomotopic in $M - F_1$. Let f_1 be a map such that $f_1|(T - \mathcal{D}) = f|(T - \mathcal{D})$ and $f_1(\mathcal{D}) \cap F_1$ is empty. Since $\pi_2(M) = 0$, f_1 is homotopic to f. The existence of f_1 contradicts the minimality condition on f so we may suppose that $f^{-1}(F_1)$ is a collection of disjoint simple essential loops.

Let $A_1, \ldots, A_n$ be the closures of the components of $T - f^{-1}(F_1)$. Our minimality condition above insures that $f|A_i$ is not homotopic rel $A_i \cap f^{-1}(F_1)$ to a map into F_1. Let R be a regular neighborhood of F_1.

Suppose for some j, $f(A_j)$ meets only one component of $R - F_1$. It is a consequence of Lemma 5.12 that there is an essential embedding $\bar{f}\colon K \longrightarrow M$ where K is a closed, connected surface of genus one. If K is a torus, we are finished. Otherwise let N be a regular neighborhood of $\bar{f}(K)$. Now N is orientable, so ∂N is a torus and $\pi_1(\partial N) \longrightarrow \pi_1(M)$ is a monic since $\bar{f}_*$ is monic. Now $\bar{f}^{-1}(F_1)$ contains a simple loop λ such that $\bar{f}(\lambda)$ is not freely homotopic to a loop in ∂F_1. Since F_1 is orientable one component of $N \cap F_1$ is an annular neighborhood of $\bar{f}(\lambda)$. It follows that ∂N is an essential torus in M since a loop in $\partial N \cap F_1$ is not freely homotopic to a loop in ∂M.

We suppose for all i where $1 \leqslant i \leqslant n$, $f(A_i)$ meets both components of $R - F_1$ and that M does not admit an essential embedding of A. It is a consequence of Lemma 5.11 that there is a surface K such that $\chi(K) = 0$ and a map $\bar{f}\colon K \longrightarrow M$ such that

(1) $\bar{f}^{-1}(F_1)$ is a collection of disjoint simple essential loops.

(2) The restriction of $\bar{f}$ to each component of $\bar{f}^{-1}(F_1)$ is a homeomorphism.

(3) The restriction of $\bar{f}$ to each component of $K - \bar{f}^{-1}(F_1)$ is a homeomorphism and the closure of each component of $K - \bar{f}^{-1}(F_1)$ is an annulus. (Note that if $\bar{f}^{-1}(F_1)$ is connected, $\bar{f}$ is an embedding and the proof is completed as above.)

If the image under $\bar{f}$ of some component of $K - \bar{f}^{-1}(F_1)$ meets only one component of $R - F_1$, we appeal to Lemma 5.12 and complete the proof of Theorem 4 as above. We suppose that $\bar{f}$ has been chosen so that the images of the loops in $\bar{f}^{-1}(F_1)$ are in general position with respect to one another on F_1. Let $\lambda_1, \ldots, \lambda_n$ be the components of $\bar{f}^{-1}(F_1)$ and

$$X(\bar{f}) = \bigcup_{i \neq j; 1 \leq i < j \leq n} (\bar{f}(\lambda_i) \cap \bar{f}(\lambda_j)).$$

We suppose that $X(\bar{f})$ is a finite set and that if $1 \leq i < j < k \leq n$, $\bar{f}(\lambda_i) \cap \bar{f}(\lambda_j) \cap \bar{f}(\lambda_k)$ is empty. Suppose that β_1 and β_2 are arcs embedded in $\bar{f}(\lambda_i)$ and $\bar{f}(\lambda_j)$ respectively such that

(a) $\partial\beta_1 = \partial\beta_2 = \beta_1 \cap \beta_2$.

(b) $\beta_1 \cup \beta_2$ bounds a disk $\mathcal{D}$ in F_1.

(c) For each arc $\beta \subset \bigcup_{i=1}^n \bar{f}(\lambda_i)$ properly embedded in $\mathcal{D}$, β meets both β_1 and β_2.

Then it can be shown that $\bar{f}$ is homotopic to a map $\bar{f}_1$ such that $X(\bar{f}_1)$ contains fewer points than $X(\bar{f})$ so that $\bar{f}_1^{-1}(F_1) = \bar{f}^{-1}(F_1)$ and $\bar{f}_1$ satisfies (1)–(3) above. Thus we may suppose that no such β_1 and β_2 exist. Let $A_1, \ldots, A_n$ be the closures of the components of $K - \bar{f}^{-1}(F_1)$. After a general position argument, we may assume that the closure of $A_i \cap \bar{f}^{-1}\bar{f}^{-1}\bar{f}(\text{int}(A_j))$ is the union of a collection of disjoint simple arcs and loops for all i, j such that $1 \leq i < j \leq n$. Suppose there are arcs α_1 and α_2 properly embedded in A_i and A_j respectively such that $\bar{f}(\alpha_1) = \bar{f}(\alpha_2)$ and α_1 and α_2 are not spanning arcs of A_i and A_j respectively. Let $\mathcal{D}_1$ be the disk on A_i cut off by α_1 and $\mathcal{D}_2$ be the disk on A_j cut off by α_2. Observe that $\bar{f}(\mathcal{D}_1) \cap \bar{f}(A_j)$ is a collection of disjoint simple arcs and loops properly embedded in the embedded disk $\bar{f}(\mathcal{D}_1)$. Thus by making appropriate choices for α_1 and α_2, we may assume that $\bar{f}(\mathcal{D}_1) \cap \bar{f}(A_j)$ contains a collection of simple loops and exactly one arc $\bar{f}(\alpha_1) = \bar{f}(\alpha_2)$. Let β_1 and β_2 be the closures of $\bar{f}(\partial\mathcal{D}_1 - \alpha_1)$ and $\bar{f}(\partial\mathcal{D}_2 - \alpha_2)$ respectively. Then $\beta_1 \cup \beta_2$ bounds a singular disk $\bar{f}(\mathcal{D}_1) \cup \bar{f}(\mathcal{D}_2)$ in M so $\beta_1 \cup \beta_2$ bounds a disk $\mathcal{D}$ on F_1. It follows from the usual argument that we may suppose that each arc in $\mathcal{D} \cap \bar{f}(K)$ properly embedded in $\mathcal{D}$ meets both β_1 and β_2. This contradicts our assumption on the minimality of $X(\bar{f})$, so we may assume that each arc in the closure of $A_i \cap \bar{f}^{-1}\bar{f}(\text{int}(A_j))$ and properly embedded in A_i is a spanning arc of A_i.

If $\bar{f}|\bar{f}^{-1}(F_1)$ is an embedding, Theorem 4 is an immediate consequence of Lemma 5.13.

Let N be a regular neighborhood of $\bar{f}(K)$. Let $G = N \cap F_1$. Let $\bar{N}$ be the manifold obtained by splitting N along G and $P: \bar{N} \longrightarrow N$ the natural identification map. Denote the copies of G in $\partial\bar{N} \cap P^{-1}(G)$ by G_1 and G_2 and observe that G_1 and G_2 need not be connected. Let f_i be $\bar{f}|A_i$ for $i = 1, \ldots, n$ and $\hat{f}_i$: $A_i \longrightarrow \bar{N}$ be the map induced by f_i. Suppose μ_1 is a simple loop in G_1. Then we claim that μ_1 is freely homotopic in $\hat{N}$ to a loop in G_2. Since $\bar{f}(K)$ is a deformation retract of N, $(\bigcup_{i=1}^n \hat{f}_i(A_i)) \cap G_1$ is a deformation retract of G_1. Thus μ_1 is freely homotopic in G_1 to a loop in $\bigcup_{i=1}^n \hat{f}_i(\partial A_i)$. If μ_1 is freely homotopic to a loop λ in $\hat{f}_i(\partial A_i)$ in G_1, μ_1 is freely homotopic in $\bar{N}$ to a loop in G_2 since λ is freely homotopic in $\hat{f}_i(A_i)$ to a loop in $\hat{f}_i(\partial A_i) - \lambda$. Let

$$X = \bigcup_{1 \leq i < j \leq m} (\hat{f}_i(\partial A_i) \cap \hat{f}_j(\partial A_j)).$$

Now μ_1 is homotopic in G_1 to a loop $\mu_2 \subset \bigcup_{i=1}^n \hat{f}_i(\partial A_i)$ where μ_2 is the union of embedded arcs $\beta_1(t), \ldots, \beta_S(t)$ for t in $[0, 1]$ such that

(1) $\beta_i \cap X = \partial\beta_i$ for $i = 1, \ldots, S$;

(2) $\beta_i(1) = \beta_{i+1}(0)$ for $i = 1, \ldots, S-1$;

(3) $\beta_1(0) = \beta_S(1)$.

Note that a point in X lying on $\hat{f}_i(A_i)$ determines a spanning arc in $\hat{f}_i(A_i)$ that lies in $\hat{f}_i(\hat{A}_i) \cap \hat{f}_j(A_j)$ where $1 \leq j \leq S$ and $j \neq i$. Thus if β_i lies on $\hat{f}_i(A_i)$, $\partial\beta_i$ determines two spanning arcs α_1 and α_2 on $\hat{f}_i(A_i)$. These spanning arcs are not necessarily disjoint. Consider the arc on $\hat{f}_i(A_i)$ obtained by running along α_1 from the component of $\partial\hat{f}(A_i)$ not meeting β_i to the component of $\partial\hat{f}(A_i)$ meeting β_i, along β_i and then along α_2 to the component of $\partial\hat{f}_i(A_i)$ not containing β_i. This arc is homotopic rel its endpoints to an arc β_i' in $\partial\hat{f}_i(A_i)$. Thus for each β_i, there is a natural way to associate a $\beta_i' \subset G_2$ for $i = 1, \ldots, S$ with β_i. Furthermore we can fit the homotopies above together in a natural way to form a free homotopy from μ_2 to $\bigcup_{i=1}^S \beta_i'$. This completes the proof of our claim.

It is a consequence of our claim above and the theorem in [14] that if λ is a simple loop in G_1, there is an annulus $\bar{A}$ embedded in $\bar{N}$ such that $\bar{A} \cap G_1 = \lambda$ and $\bar{A} \cap G_2 = \partial\bar{A} - \lambda$.

Note that each component of ∂G which is nullhomotopic in M bounds a disk on F_1. Let G^* be the submanifold of F_1 obtained by attaching all such disks to G. Now any loop in G^* that is nullhomotopic in F_1 is nullhomotopic in G^*. Since $\bar{f}^{-1}(G^*)$ contains an essential loop, no component of G^* is a disk. If a component G_1^* of G^* is an annulus, $\bar{f}$ carries the components of $\bar{f}^{-1}(G_1^*)$ to disjoint loops and as a consequence of Lemma 5.13 $\bar{f}$ may be taken to be an embedding. Now Theorem 4 follows as above. If G^* is planar and has exactly three boundary components $\bar{f}|\bar{f}^{-1}(G^*)$ may be taken to be an embedding. It is a consequence of Lemma 5.13 that $\bar{f}$ could be taken to be an embedding and Theorem 4 would follow as above.

We suppose then that if a component of G^* is planar it has at least four boundary components. Let $\lambda_1, \ldots, \lambda_r$ be a collection of essential simple loops properly embedded in G such that

(1) Arcs $\beta_1 \subset \lambda_i$ and $\beta_2 \subset \lambda_j$ do not exist where $\beta_1 \cup \beta_2$ bounds a disk on G^* for $1 \leq i < j \leq r$.

(2) $G^* - (\bigcup_{i=1}^r \lambda_i \cup \partial G^*)$ is a collection of open annuli (or a disk if G^* is closed).

(3) $\bigcup_{i=1}^{r}\lambda_i$ is a deformation retract of G^* if G^* is not closed.

(4) $\lambda_i \cap \lambda_j$ contains at most two points for $1 \leqslant i < j \leqslant r$.

The reader may observe that $\lambda_i \cap \lambda_j$ for $1 \leqslant i < j \leqslant r$ may be assumed to be a single point if G^* is not planar.

Let $\bar{A}_1, \ldots, \bar{A}_r$ be a collection of annuli properly embedded in $\bar{N}$ such that

(1) $P(\bar{A}_i \cap G_1) = \lambda_i$ for $i = 1, \ldots, r$.

(2) $\bar{A}_i \cap G_2$ is a loop in $\mathrm{int}(G_2)$ for $i = 1, \ldots, r$.

After the usual cutting argument, we may suppose that $\bar{A}_i \cap \bar{A}_j$ is a collection of disjoint spanning arcs for $1 \leqslant i < j \leqslant r$.

Let N_1 be a regular neighborhood of $G^* \cup \bigcup_{i=1}^{r} P(\bar{A}_i)$. It is not difficult to see that if one splits N_1 along $F_1 \cap N_1$, one obtains a 3-manifold homeomorphic to $G^* \times [0, 1]$ since G^* is a deformation retract of $N_1 \cap F_1$. Let G_1^* be a component of G^*. Then it can be seen that N_1 is a bundle with base S^1 and fibre G_1^*. Since N_1 is an orientable 3-manifold, ∂N_1 is an orientable 2-manifold. Since N_1 is a bundle with base S^1 and fibre G_1^*, ∂N_1 is a collection of tori. Since ∂M is not empty, ∂N_1 is not empty. Let T_1 be a component of ∂N_1. We claim T_1 is incompressible in M. Let λ_1 be a simple loop in $T_1 \cap F_1$. Since $\pi_1(G_1^*) \longrightarrow \pi_1(F_1)$ is monic, λ_1 is essential in M. Let λ_2 be a simple loop on T_1 that meets each loop in $T_1 \cap F_1$ in a single point and crosses F_1 at each point in $\lambda_2 \cap F_1$. Then the intersection number of λ_2 and F_1 can be seen to be the same as the cardinality of $F_1 \cap \lambda_2$. Suppose $[\lambda] \in \pi_1(T_1)$ is an element such that λ is an inessential loop in M. Then $[\lambda] = s_1[\lambda_1] + s_2[\lambda_2]$. Since the intersection number of λ_1 and F_1 is not zero and that of λ_2 and F_1 is zero, $s_1 = 0$. Since λ_2 is not nullhomotopic on F_1 and F_1 is incompressible in M, $s_2 = 0$. Thus $[\lambda] = 0 \in \pi_1(T_1)$ and T_1 is incompressible in M. If T_1 is not parallel to a component of ∂M, T_1 is an essential embedding of a torus in M and Theorem 4 follows.

Otherwise it can be seen that N_1 is a deformation retract of M and M may be taken to be a bundle with base S^1 and fibre F_1. It follows that G^* is connected as is the regular neighborhood in F_1 of $F_1 \cap \bar{f}(K)$. Denote such a neighborhood by F. Then it can be seen that $\pi_1(F) \longrightarrow \pi_1(F_1)$ is an epimorphism.

We claim that M has a finite sheeted cyclic covering $(\widetilde{M}, p)$ homeomorphic to $F_1 \times S^1$. If this is true, there is an essential map $f\colon (A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ since F_1 is not a disk. But then $pf\colon (A, \partial A) \longrightarrow (\widetilde{M}, \partial\widetilde{M})$ is an essential map and Theorem 4 will follow from Theorem 2.

The proof of Theorem 4 will be complete when we establish the claim above. Let κ be the number of loops in $\bar{f}^{-1}(F_1)$. Let (M^*, q) be the κ-sheeted cyclic covering of M associated with F_1. Then there is an embedding $\bar{g}_1\colon K \longrightarrow M^*$ such that $q\bar{g}_1 = \bar{f}$. Let $\rho^*\colon M^* \longrightarrow M^*$ be a generator of the group of covering translations. Now $\rho^{*i}g_1(K) \cap \rho^{*j}g_1(K)$ is the union of a collection of disjoint simple loops for $0 \leq i < j < \kappa$ since the closure of $A_i \cap \bar{f}^{-1}\bar{f}(\text{int}(A_j))$ is a collection of disjoint simple spanning arcs of A_i and simple loops in A_i for $1 \leq i < j \leq \kappa$.

Let $\bar{F}_1$ be a component of $p^{-1}(F_1)$. Let κ_{ij} be the intersection number of $\bar{F}_1$ and any loop in $\rho^{*i}g_1(K) \cap \rho^{*j}g_1(K)$ that meets $\bar{F}_1$. If no loop in $\rho^{*i}g_1(K) \cap \rho^{*j}g_1(K)$ meets F_1, let $\kappa_{ij} = 1$ for $0 \leq i < j < \kappa$. Let $\tilde{\kappa}$ be the least common multiple of the κ_{ij} and 2.

Let $(\tilde{M}, p_1)$ be the $\tilde{\kappa}$-sheeted cyclic covering space of M^* associated with $\bar{F}_1$. Then we claim there is an embedding $h_i\colon T \longrightarrow \tilde{M}$ such that $p_1 h_i(T) = \rho^{*i}\bar{g}_1(K)$ and $(h_i(T), p|h_i(T))$ is a $\tilde{\kappa}$-sheeted cyclic covering of $\rho^{*i}\bar{g}_1(K)$ for $0 \leq i < \kappa$. Clearly we need only show that $p_1^{-1}\bar{g}_1(K)$ is a torus. Since, for each simple loop $\lambda \subset \bar{g}_1^{-1}q^{-1}(F_1)$, $K - \lambda$ is an open annulus, each nonorientable loop on K has odd intersection number with λ. Thus $p_1^{-1}\bar{g}_1(K)$ is a torus.

Let $p = qp_1$. Then $(\tilde{M}, p)$ is a $\kappa * \tilde{\kappa}$-sheeted cyclic covering of M. Let $\tilde{F}_1$ be a component of $p^{-1}(F_1)$. Note that each simple loop in $h_i(T) \cap h_j(T)$ meets F_1 in at most a single point for $0 \leq i < j < \kappa$. Split $\tilde{M}$ along $\tilde{F}_1$ to obtain a 3-manifold $\tilde{N}$. Let $P\colon \tilde{N} \longrightarrow \tilde{M}$ be the natural identification map and $\hat{h}_i\colon (A, \partial A) \longrightarrow (\tilde{N}, \partial\tilde{N})$ the maps induced by the $h_i\colon T \longrightarrow \tilde{M}$ for $0 \leq i < \kappa$.

We suppose that for each i where $0 < i < \kappa$ there is a $j < i$ where $0 \leq j$ such that $\hat{h}_i(A) \cap \hat{h}_j(A)$ contains a spanning arc of $\hat{h}_i(A)$. This is possible since $\bigcup_{i=0}^{\kappa}(h_i(T) \cap \tilde{F}_1)$ is connected although it may be necessary to reorder the subscripts on the h_i for $0 \leq i < \kappa$.

We have that $\hat{h}_0(A) \cap \hat{h}_1(A)$ is a collection of disjoint simple loops and spanning arcs. Using standard techniques, we can find proper embeddings $\hat{h}'_0\colon A \longrightarrow \tilde{N}$ and $\hat{h}'_1\colon A \longrightarrow \tilde{N}$ such that

(1) $\hat{h}'_j(\partial A) = \hat{h}_j(\partial A)$ for $j = 0, 1$.

(2) $\hat{h}'_0(A) \cap \hat{h}'_1(A)$ is the union of a collection of disjoint simple spanning arcs.

(3) $\hat{h}'_0(A) \cap \hat{h}'_1(A) \subset \hat{h}_0(A) \cap \hat{h}_1(A)$.

Since $\hat{h}'_0(A)$ and $\hat{h}_0(A)$ differ only on the interior of the union of a collection of disjoint disks on A and $\pi_2(\tilde{N}) = 0$, $\hat{h}'_0(A)$ and $\hat{h}_0(A)$ are homotopic rel $\hat{h}_0(\partial A)$. Similarly $\hat{h}_1(A)$ and $\hat{h}'_1(A)$ are homotopic rel $\hat{h}_1(\partial A)$. Thus if $2 \leq i < \kappa$ and $\hat{h}'_i(A)$ is homotopic rel $\partial\hat{h}'_i(A)$ to $\hat{h}_i(A)$ and if α_1 is a spanning arc of $\hat{h}'_i(A)$ in $\hat{h}'_i(A) \cap \hat{h}'_0(A)$, α_1 is homotopic rel its endpoints to a spanning arc in $\hat{h}_i(A) \cap \hat{h}_0(A)$. A similar statement holds for a spanning arc $\hat{h}'_i(A)$ in $\hat{h}'_i(A) \cap \hat{h}'_1(A)$.

Let N_1 be a regular neighborhood of $h'_0(A) \cup \hat{h}'_1(A)$ in $\tilde{N}$. Let $\bar{G}_1$ be a regular neighborhood of $(h_0(T) \cup h_1(T)) \cap \tilde{F}_1$ in $\tilde{F}_1$. We may suppose that $P^{-1}\bar{G}_1 = N_1 \cap P^{-1}(\tilde{F}_1)$. Since $\hat{h}'_0(A) \cap \hat{h}'_1(A)$ is a collection of disjoint simple spanning arcs of $\hat{h}'_0(A)$ and $\hat{h}'_1(A)$, there is a homeomorphism $\bar{\theta}_1\colon \bar{G}_1 \times [0, 1]$

$\longrightarrow N_1$ such that

(1) $\bar{\theta}_1(\bar{G}_1 \times [0, 1]) = N_1$.

(2) $\bar{\theta}_1(\bar{G}_1 \times \{0, 1\}) = P^{-1}\bar{G}_1$.

(3) There are simple loops μ_0 and μ_1 on $\bar{G}_1$ such that $\bar{\theta}_1(\mu_0 \times [0, 1]) = \hat{h}'_0(A)$ and $\bar{\theta}_1(\mu_1 \times [0, 1]) = \hat{h}'_1(A)$.

(4) $P\bar{\theta}_1(x, 0) = P\bar{\theta}_1(x, 1)$ for $x \in \bar{G}_1$.

Suppose that $\pi_1(\bar{G}_1) \longrightarrow \pi_1(\widetilde{F}_1)$ is not monic. Then there is a disk $\mathcal{D}_1$ embedded in $P^{-1}(\widetilde{F}_1)$ such that $\mathcal{D}_1 \cap P^{-1}(\bar{G}_1) = \partial\mathcal{D}_1$. Thus there is a component a_1 of $\partial\bar{G}_1$ such that $\bar{\theta}_1(a_1 \times \{j\}) = \partial\mathcal{D}_1$ for $j = 0$ or 1. We suppose $\bar{\theta}_1(a_1 \times \{0\}) = \partial\mathcal{D}_1$. Since $\bar{\theta}_1(a_1 \times [0, 1]) \cup \mathcal{D}_1$ is an embedded disk and $P^{-1}\widetilde{F}_1$ is incompressible, $\bar{\theta}_1(a_1 \times \{1\})$ bounds a disk $\mathcal{D}_2$ embedded in $P^{-1}(\widetilde{F}_1)$. Now $\mathcal{D}_2 \cap P^{-1}(\bar{G}_1) = \partial\mathcal{D}_2$ and $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \bar{\theta}_1(a_1 \times [0, 1])$ is a 2-sphere S^2 embedded in $\widetilde{N}$. It is a consequence of Theorem 8.1 in [15] that $\widetilde{N}$ is irreducible. Thus S^2 bounds a 3-ball B^3 embedded in $\widetilde{N}$. It follows that $\bar{\theta}_1$ can be extended to a homeomorphism of $(\bar{G}_1 \cup P(\mathcal{D}_1)) \times [0, 1]$ onto $N_1 \cup B^3$. Since $\bar{G}_1$ has only finitely many boundary components, we may suppose that there is a surface G_1 embedded in F_1 such that $G_1 \supseteq \bar{G}_1$ and $\pi_1(G_1) \longrightarrow \pi_1(F_1)$ is monic. Applying the argument given above inductively, we may suppose that there is an embedding $\theta_1\colon G_1 \times [0, 1] \longrightarrow \widetilde{N}$ that extends $\bar{\theta}_1$ such that

(1) $\theta_1(G_1 \times \{0, 1\}) = P^{-1}G_1$.

(2) There are loops μ_0 and μ_1 on G_1 such that $\theta_1(\mu_0 \times [0, 1]) = \hat{h}'_0(A)$ and $\theta_1(\mu_1 \times [0, 1]) = \hat{h}'_0(A)$.

(3) $P\theta_1(x, 0) = P\theta_1(x, 1)$ for $x \in G_1$.

Note that $\theta_1(\partial G_1 \times [0, 1])$ is a collection of disjoint incompressible annuli in $\widetilde{N}$.

Suppose that $\mu_0, \ldots, \mu_k, G_k$, and θ_k have been defined as above for $k < \kappa$. After an argument similar to those given above, we may assume that $\hat{h}'_{k+1}(A) \cap \theta_k(\partial G_k \times [0, 1])$ is a nonempty collection of disjoint simple arcs. Thus $\hat{h}'_{k+1}(A) \cap \theta_k(G_k \times [0, 1])$ is a nonempty collection of disks properly embedded in $\theta_k(G_k \times [0, 1])$.

Note that if $\hat{h}'_i(c_1)$ is freely homotopic to a component of $\theta_k(\partial G_k \times \{0, 1\})$ in $\theta_k(\partial G_k \times \{0, 1\})$, the loop $P\hat{h}'_i(c_1) = h_i(T) \cap \widetilde{F}_1$ is isotopic in $\widetilde{F}_1$ to a loop that fails to meet $h_j(T) \cap \widetilde{F}_1$ for $j \neq i$, $0 \leqslant j \leqslant k$. This implies that there are arcs β_1 and β_2 in $\bar{f}(K) \cap F_1$ such that $\beta_1 \cap \beta_2 = \partial\beta_1 = \partial\beta_2$, $\beta_1 \cup \beta_2$ bounds a disk on F_1 and β_1 (β_2) lies in the image of a component of $\bar{f}^{-1}(F_1)$. This contradicts an earlier hypothesis.

We proceed with the assumption that $\hat{h}'_i(c_1)$ and $\hat{h}'_i(c_2)$ are not freely homotopic in $\widetilde{F}_1$ to loops in $\partial P^{-1}G_k$ for $0 \leqslant i \leqslant k$. We may also assume that each disk $\mathcal{D}$ in $\hat{h}'_{k+1}(A) \cap \theta_k(G_k \times [0, 1])$ contains a spanning arc of $h'_{k+1}(A)$ and one of the annuli $\hat{h}'_i(A)$ where $0 \leqslant i \leqslant k$. Suppose $\mathcal{D} \cap \hat{h}'_0(A)$ contains a spanning arc of $\hat{h}'_{k+1}(A)$. We show that $\mathcal{D}$ is isotopic rel $\partial\mathcal{D} \cap \theta_k(G_k \times \{0, 1\})$ to a product disk i.e., $\beta \times [0, 1]$ where β is an arc properly embedded in G_k. We may assume that $\mathcal{D} \cap \hat{h}'_0(A)$ is the union of a collection of simple spanning arcs of $\hat{h}'_0(A)$. Let α_1 be a spanning arc in this collection. Let $\mathcal{D}_1$ be the closure of one component of $\mathcal{D} - \alpha_1$. We may suppose that $\mathcal{D}_1 \cap \hat{h}'_0(A) = \alpha_1$. Now α_1 is homotopic rel its endpoints to a spanning α'_1 in $\hat{h}_0(A) \cap \hat{h}_{k+1}(A)$ such that $P(\partial\alpha'_1)$ is a single point. Thus there is a point x_1 in G_k such that $\theta_k(\{x_1\} \times \{0, 1\}) = \partial\alpha_1$.

We will show that the arc α_1 is homotopic rel its endpoints to $\theta_k(\{x_1\} \times [0, 1])$. Let $\phi_k: G_k \times [0, 1] \longrightarrow G_k$ be defined by $\phi_k(x, t) = x$ for x in G_k, t in $[0, 1]$. Let $\beta_0 = \mathcal{D} \cap \theta_k(G_k \times \{0\})$ and $\beta_1 = \mathcal{D} \cap \theta_k(G_k \times \{1\})$. Now $\phi_k\theta_k^{-1}(\beta_0) = \phi_k\theta_k^{-1}(\beta_1)$ since $P(\beta_0) = P(\beta_1)$ for $P\theta_k(\{x_1\} \times \{0\}) = P\theta_k(\{x_1\} \times \{1\})$. Thus $\phi_k\theta_k^{-1}: \mathcal{D} \longrightarrow G_k$ determines a map of either an annulus or a möbius band into G_k. Since $P\hat{h}'_{k+1}(A)$ is orientable, we see that $\phi_k\theta_k^{-1}: \mathcal{D} \longrightarrow G_k$ determines a map $\epsilon_k: (A, \partial A) \longrightarrow (G_k, \partial G_k)$.

Consider the map $\bar{\epsilon}_k: A \longrightarrow G_k$ induced by $\phi_k\theta_k^{-1}|\mathcal{D}_1$. If $\bar{\epsilon}_{k*}$ is monic, μ_0 is homotopic to a loop in ∂G_k which we have shown leads to a contradiction. It follows that $\bar{\epsilon}_k(c_1)$ and $\bar{\epsilon}_k(c_2)$ are nullhomotopic in G_k. Thus the loop $\phi_k\theta_k^{-1}(\mathcal{D}_1 \cap \hat{h}'_i(A))$ is nullhomotopic and the arc α_1 is homotopic rel its boundary to $\theta_1(\{x_1\} \times [0, 1])$ as was to be shown.

It follows that after a homotopy of $\hat{h}'_{k+1}$ rel ∂A, we may suppose that $\theta_k^{-1}(\hat{h}'_{k+1}(A))$ is a collection of product disks i.e. $\phi_k\theta_k^{-1}(\hat{h}'_{k+1}(A))$ is a collection of arcs. But now θ_k can be extended in an obvious way to $\theta_{k+1}: G_{k+1} \times [0, 1] \longrightarrow \widetilde{N}$ where $\widetilde{F}_1 \supseteq G_{k+1} \supseteq G_k$ so that

(1) $\theta_{k+1}(G_{k+1} \times \{0, 1\}) = P^{-1}(G_{k+1})$.

(2) There are loops $\mu_0, \ldots, \mu_{k+1}$ on G_{k+1} such that $\theta_{k+1}(\mu_i \times [0, 1]) = \hat{h}'_i(A)$ for $i = 1, \ldots, k+1$.

(3) $P\theta_{k+1}(x, 0) = P\theta_{k+1}(x, 1)$ for $x \in G_{k+1}$.

(4) $\pi_1(G_{k+1}) \longrightarrow \pi_1(\widetilde{F}_1)$ is monic.

It follows that there is an embedding $\theta_\kappa: G_\kappa \times [0, 1] \longrightarrow \widetilde{N}$ such that

(1) $P\theta_\kappa(x, 0) = P\theta_\kappa(x, 1)$ for $x \in G_\kappa$.

(2) $P\theta_\kappa(G_\kappa \times \{0\}) \supseteq \bigcup_{i=0}^{\kappa-1} h_i(T) \cap \widetilde{F}_1$.

(3) $\theta_{\kappa*}$ is monic.

Since $\theta_\kappa(G_\kappa \times \{0, 1\}) \supseteq \bigcup_{i=0}^{\kappa-1} \hat{h}'_i(\partial A)$, $\theta_{\kappa*}$ is onto. Thus $P\theta_\kappa(G_\kappa \times \{0\}$ is a deformation retraction of $\widetilde{F}_1$. But it can now be seen that $P\theta_\kappa(G_\kappa \times [0, 1])$ is homeomorphic to $G_\kappa \times S^1$ and is a deformation retraction of $\widetilde{M}$. It follows that $\widetilde{M}$ is homeomorphic to $G_\kappa \times S^1$ and Theorem 4 follows.

VII. Other theorems and applications.

THEOREM 5. *Let M be a compact, bounded, irreducible orientable 3-manifold. If M admits an S-essential map, M admits an essential embedding $h: T \longrightarrow M$.*

PROOF. Let $f: T \longrightarrow M$ be an S-essential map. By Theorem 4, M admits an essential embedding $g_1: (A, \partial A) \longrightarrow (M, \partial M)$ or an essential embedding $h: T \longrightarrow M$. Assume the former and that f is not homotopic to a map f_1 such that $f_1(T) \cap g_1(A)$ is empty. It is a consequence of Lemma 5.6 that there is an embedding $h: T \longrightarrow M$ such that $h^{-1}(T)$ contains a simple loop λ so that $h(\lambda)$ is freely homotopic in $g_1(A)$ to a loop in $\partial g_1(A)$ and h_* is monic. We may suppose that λ_1 is a simple essential loop in $f^{-1}(g_1(A))$. Since λ_1 is essential and f is S-essential $f(\lambda_1)$ is not freely homotopic to a loop in ∂M. Since $f(\lambda_1)$ is freely homotopic to a loop in $g_1(\partial A)$, $f(\lambda_1)$ is freely homotopic to a multiple of $h(\lambda)$ and $h(\lambda)$ is not freely homotopic to a loop in ∂M. Thus h is essential.

If f is homotopic to a map f_1 such that $f_1(T) \cap g_1(A)$ is empty, we trade f for f_1 and assume $f(T) \cap g_1(A)$ is empty. Let M_1 be the manifold obtained by removing the interior of a regular neighborhood of $g_1(A)$ from M. If $f: T \longrightarrow M_1$ is essential, we repeat the argument above. As a result of this argument, either we find an essential embedding $g_2: (A, \partial A) \longrightarrow (M_1, \partial M_1)$ or an embedding $h: T \longrightarrow M_1$ such that $h: T \longrightarrow M$ is essential. In case we find $g_2: A \longrightarrow M_1$ and f is homotopic to a map f_1 such that $f_1(T) \cap g_2(A)$ is empty, we suppose that $f(T) \cap g_2(A)$ is empty and let M_2 be the 3-manifold obtained from M_1 by removing the interior of a regular neighborhood of $g_2(A)$ from M_1. If f is not homotopic to a map such that $f(T) \cap g_2(A)$ is empty, we find a map $h: T \longrightarrow M_1$ as above so that $h: T \longrightarrow M$ is essential.

We continue the process above until either we have found an essential map $h: T \longrightarrow M$ or $f: T \longrightarrow M_i \subset M$ is inessential in M_i. Note that we have constructed a partial hierarchy for M and the process above must terminate.

Suppose ∂M_i is not incompressible in M_i. Then there is a disk $\mathcal{D}$ properly embedded in M_i such that $\partial \mathcal{D}$ is essential in ∂M_i. We may suppose that $f^{-1}(\mathcal{D})$ is the union of a collection of simple loops. If λ in $f^{-1}(\mathcal{D})$, λ is nullhomotopic on T since f_* is monic. Thus after the usual argument we may suppose that $f(T) \cap \mathcal{D}$ is empty. It can now be seen that we can assume that ∂M_i is incompressible and M_i is irreducible. It follows from standard arguments that f is homotopic to a map into ∂M_i. Suppose that $f_1: T \longrightarrow \partial M_i$ is homotopic to f. Let $T_1 = f_1(T)$.

Note that $f_1: T \longrightarrow T_1$ is homotopic to a covering map by Lemma 1.4.3 in [15] since $f_{1*}: \pi_1(T) \longrightarrow \pi_1(T_1)$ is monic and that T_1 is a torus. Let $h: T \longrightarrow T_1$ be an embedding. Then $h_*: \pi_1(T) \longrightarrow \pi_1(M)$ is monic since ∂M_i is incompressible in M_i (and in M). It can now be seen that h is an essential map. This completes the proof of Theorem 5.

THEOREM 6. *Let M be a compact, orientable, irreducible, boundary irreducible 3-manifold. Let the torus T_1 be a component of ∂M. Suppose M admits an essential embedding $g: (A, \partial A) \longrightarrow (M, T_1)$. If $g(A)$ separates M, we denote* *the closures of the components of $M - g(A)$ by M_1 and M_2. Then either M admits an essential embedding of T or $g(A)$ separates M and M_j is either a solid torus or homeomorphic to $T \times [0, 1]$ for $j = 1, 2$.*

PROOF. Let A_1 and A_2 be the closures of the components of $T_1 - g(\partial A)$. Let $g_j\colon T \longrightarrow M$ be an embedding such that $g_j(T)$ is parallel to the torus $A_j \cup g(A)$ for $j = 1, 2$. If g_1 is not essential, either $g_{1*}\colon \pi_1(T) \longrightarrow \pi_1(M)$ is not monic or $g_1(T)$ is parallel to a component of ∂M. In the former case, the usual argument shows $g_1(T)$ bounds a solid torus N in M. Since T_1 is incompressible in M, T_1 is not contained in N. Thus $g(A)$ is not contained in N. This shows that M_1 may be taken to be a solid torus.

In the latter case $g_1(T)$ and a component F of ∂M bound a submanifold N of M homeomorphic to $T \times [0, 1]$. Since g is essential, $F \neq T_1$ and $g(A)$ is not contained in N. Since $g_1(T)$ is parallel to $A_1 \cup g(A)$, the desired manifold M_1 is the union of N with the submanifold of M bounded by $g_1(T)$ and $A_1 \cup g(A)$. We observe that $\partial M_1 = (\partial N - g_1(T)) \cup A_1 \cup g(A)$.

A similar argument establishes the existence of the desired M_2. This completes the proof of Theorem 6.

COROLLARY TO THEOREM 6. *Let M be the knot space of a knot K. If M admits an essential map of a torus but not an essential embedding of a torus, K is a torus knot.*

PROOF. This result follows immediately from Theorem 4 and Theorem 6.

We observe that Theorem 7 below gives a partial answer to question T on p. 101 in [8]. Theorem 7 also seems to be related to problem 4, p. 168 in [7]. This question has been partially answered in [4].

THEOREM 7. *Let M be the space of a knot K. Suppose $\pi_1(M)$ has a subgroup A isomorphic to $Z \oplus Z$ such that A is not conjugate to a subgroup of a peripheral subgroup of $\pi_1(M)$. Then either K is a torus knot or M admits an essential embedding $g\colon T \longrightarrow M$ such that $g_*\pi_1(T)$ is not conjugate to a subgroup of a peripheral subgroup of $\pi_1(M)$.*

PROOF. Let $f\colon T \longrightarrow M$ be a map such that $f_*\pi_1(T) = A \subseteq \pi_1(M)$. Note that f is essential since if f were homotopic to a map into ∂M, A would be conjugate to a subgroup of a peripheral subgroup of $\pi_1(M)$. Thus by Theorem 6 either K is a torus knot or M admits an essential embedding $g\colon T \longrightarrow M$. But since $g(T)$ is not parallel to ∂M, $g_*\pi_1(T)$ is not conjugate to a subgroup of $\pi_1(\partial M) \subseteq \pi_1(M)$. This completes the proof of Theorem 7.

In Theorem 8 below we give a positive answer to question 3 on p. 24 in [9] in case M is a bounded, irreducible, orientable 3-manifold and the surface in question has genus one. An example in [13] shows that Theorem 8 does not hold if either M is not required to be bounded or M is not irreducible.

THEOREM 8. *Let M be a compact, bounded, irreducible, orientable 3-manifold. Let $f: T \longrightarrow M$ be a map such that $f_*: \pi_1(T) \longrightarrow \pi_1(M)$ is monic. Then there exists an embedding $g: T \longrightarrow M$ such that $g_*: \pi_1(T) \longrightarrow \pi_1(M)$ is monic.*

PROOF. Let $\mathcal{D}$ be a disk properly embedded in M. After a homotopy we may suppose that $f^{-1}(\mathcal{D})$ is the union of a collection of disjoint simple loops. Since f_* is monic, each loop in $f^{-1}(\mathcal{D})$ is inessential on T. Thus we may suppose that $f^{-1}(\mathcal{D})$ is empty and ∂M is incompressible.

If f is S-essential Theorem 8 is a consequence of Theorem 5.

If f is homotopic to a map into ∂M, one component of ∂M is an incompressible torus embedded in M and Theorem 8 follows.

Thus we may suppose that f is W-essential. It is a consequence of Theorem 4 that M admits an essential embedding of either a torus or an annulus. In case of the former, Theorem 8 follows. Otherwise we assume that f is in general position with respect to an essential annulus A_1 in M. If f is not homotopic to a map f_1 such that $f_1(T) \cap A_1$ is empty Theorem 8 is a consequence of Lemma 5.6. Otherwise we assume that $f(T) \cap A_1$ is empty and remove the interior of a regular neighborhood of A_1 from M to obtain a 3-manifold M_1.

If $f: T \longrightarrow M_1$ is homotopic to a map into ∂M_1, Theorem 8 follows as above. Otherwise we repeat the argument above to obtain an essential embedding A_2 of an annulus in M_1. It is a consequence of the theorem on p. 60 in [15] that the construction above can be carried out at most finitely many times. Theorem 8 follows.

THEOREM 9. *Let M and N be compact, connected, bounded, irreducible 3-manifolds. Let $\Phi: \pi_1(M) \longrightarrow \pi_1(N)$ be a monomorphism. Suppose that ∂M is an incompressible torus and that M admits no essential embedding of an annulus. Then there is a submanifold N_1 of N and a covering map $\psi: M \longrightarrow N_1$ such that $\psi_* = \Phi: \pi_1(M) \longrightarrow \pi_1(N)$.*

REMARK. The following example shows that Theorem 9 is false if M is allowed to admit essential embeddings of annuli: Let p, q and r be integers and p and qr relatively prime. Let M be the space of a p, q torus knot and N the space of a p, qr torus knot. Then $\pi_1(M) = \langle x, y: x^p = y^q \rangle$ and $\pi_1(N) = \langle w, z: w^p = z^{qr} \rangle$. We define a monomorphism $\Phi: \pi_1(M) \longrightarrow \pi_1(N)$ by setting $\Phi(x) = w$ and $\Phi(y) = z^r$ and extending in the natural way. We observe that $\Phi(\pi_1(M))$ is not of finite index in $\pi_1(N)$ since $\langle x, y: x^p = y^q, y = 1 \rangle$ is a finite group while $\langle w, z: w^p = z^{qr}, z^r = 1 \rangle$ is isomorphic to the free product of the integers mod p with integers mod r which is not finite. Thus since M is compact, M is not a finite sheeted covering of N. Schubert has shown in [10] that a torus knot has no companions. Thus if Theorem 9 held, M would be a finite sheeted covering space of N. This is impossible.

PROOF OF THEOREM 9. Using standard techniques, we can find a map $f: M \longrightarrow N$ such that $f_* = \Phi$ since $\pi_2(N) = 0$. Suppose $\mathcal{D}$ is a disk properly embedded in N such that $\partial \mathcal{D}$ is essential in ∂N. Then f is homotopic to a map f_1 such that $f_1^{-1}(\mathcal{D})$ is the union of a (possibly empty) collection of disjoint disks properly embedded in M by Lemma 1.1 in [3]. Since ∂M is incompressible and $\pi_2(M) = 0$, we may assume that $f_1^{-1}(\mathcal{D})$ is empty. We replace f by f_1. Thus we may suppose that $f(M)$ lies in the complement of the interior of a regular neighborhood of $\mathcal{D}$. It follows that we may suppose that ∂N is incompressible.

Suppose A is an essential annulus properly embedded in N. As above we may assume that $f^{-1}(A)$ is a collection of disjoint annuli properly embedded in M. Since M admits no essential annuli, these annuli are parallel to annuli in ∂M. It follows that we may suppose $f^{-1}(A)$ is empty and thus that N admits no essential embeddings of annuli.

If $f: \partial M \longrightarrow N$ is homotopic to a map into ∂N, we may suppose that $f(\partial M) \subset \partial N$. But then by Theorem 6.1 in [15], f is homotopic to a covering map since M is not homeomorphic to $T \times [0, 1]$. Since this would establish Theorem 9, we may assume that $f: \partial M \longrightarrow N$ is an essential map. By Theorem 4, N admits an essential embedding $g_1: T \longrightarrow N$ or $h: (A, \partial A) \longrightarrow (N, \partial N)$. We have assumed that such an h does not exist. Let $g_i: T \longrightarrow N$ for $i = 1, \dots, n$ be a maximal collection of essential embeddings such that

(1) $g_i(T) \cap g_j(T)$ is empty for $1 \leqslant i < j \leqslant n$.

(2) $g_i(T)$ and $g_j(T)$ are not parallel for $1 \leqslant i < j \leqslant n$.

The hierarchy theorem guarantees the finiteness of this collection.

We assume that $f: M \longrightarrow N$ is transverse with respect to $\bigcup_{i=1}^n g_i(T)$. It is a consequence of Lemma 1.1 in [3] that we may take $f^{-1}(\bigcup_{i=1}^n g_i(T)) = F$ to be an incompressible surface. It can be seen as above that we may suppose no component of F is an annulus. Thus F is the union of a collection of tori.

Suppose T_1 is a component of F and T_1 is parallel to ∂M. Then f is homotopic to a map $f_1: (M, \partial M) \longrightarrow (N, f(T_1))$ so that $f_1^{-1}f(T_1)$ is the union of a nonempty collection of disjoint incompressible tori. We let $f = f_1$ without loss of generality.

If $f^{-1}f(\partial M) = \partial M$, we apply Theorem 6.1 in [15] as above to complete the proof of Theorem 9. Note that the torus $f(\partial M)$ separates N since it represents a boundary of a chain in $C_3(N; Z_2)$. Suppose $f^{-1}f(\partial M)$ is not connected. Let N_1 be the closure of a component of $N - f(\partial M)$ that contains a component of ∂N. Let M_1 be a component of $f^{-1}(N_1)$. Now $f|M_1: (M_1, \partial M_1) \longrightarrow (N_1, \partial N_1)$. Since $f(\partial M_1)$ does not meet ∂N, there is no homotopy $f_t: (M_1, \partial M_1) \longrightarrow (N_1, \partial N_1)$ such that $f_0 = f|M_1$ and f_1 is a covering map. Thus by Theorem 6 in [15], there is a homotopy as above such that $f_1(M_1) \subseteq \partial N_1$. It follows from standard arguments that we may suppose that $f^{-1}f(\partial M) = \partial M$ and Theorem 9 follows.

We suppose that $f(\partial M) \cap \bigcup_{i=1}^n g_i(T)$ is empty and $f\colon \partial M \longrightarrow N$ is not homotopic to a map into $g_i(T)$ where $1 \leqslant i \leqslant n$ and show this leads to a contradiction. This will complete the proof of Theorem 9. It is a consequence of Lemma 5.4 that $g_i\colon T \longrightarrow N$ is an S-essential map for $1 \leqslant i \leqslant n$ since it has been supposed that N does not admit an essential map of an annulus. Let N_2 be a regular neighborhood of $\bigcup_{i=1}^n g_i(T)$ in N and $N_3 = N - \operatorname{int}(N_2)$. Let N_1 be the component of N_3 that contains $f(\partial M)$. By assumption $f\colon \partial M \longrightarrow N_1$ is an essential map. Thus by Theorem 4 there is an essential embedding $g_{n+1}\colon T \longrightarrow N_1$ or an essential embedding $h\colon (A, \partial A) \longrightarrow (N_1, \partial N_1)$. Since g_i, $i = 1, \ldots, n$, is a maximal collection of essential embeddings g_{n+1} does not exist. We show below that it is impossible for h to exist.

Note that $h(\partial A)$ does not lie on ∂N since N admits no essential embeddings of annuli. If h carries one component of ∂A to ∂N and the other to $\partial N_1 - \partial N$, one of the g_i is not S-essential so we may suppose that $h(\partial A) \subset \partial N_1 - \partial N$.

If $h(\partial A)$ is contained in a single component of $\partial N_1 - \partial N$, it is a consequence of Theorem 6 that a representative of the generator of $\pi_1(h(A))$ is freely homotopic to a loop in ∂N. Thus an essential loop on $\partial N_1 - \partial N$ is freely homotopic to a loop in ∂N. This is impossible since g_i is S-essential for $i = 1, \ldots, n$.

We suppose that $h(c_1)$ and $h(c_2)$ lie on distinct components of $\partial N_1 - \partial N$. We denote these components by T_1 and T_2 respectively. Suppose that f is homotopic to a map f_1 such that $f_1(T) \cap h(A)$ is empty. Let $\hat{N}_2$ be a regular neighborhood of $T_1 \cup T_2 \cup g(A)$ in N_1. Let $\hat{N}_1$ be the closure of $N_1 - \hat{N}_2$. Now $\hat{N}_1 \cap \hat{N}_2$ is a torus T_0. Suppose T_0 is not essential in N_1. If $\pi_1(T_0) \longrightarrow \pi_1(N)$ is not monic, the usual argument shows that T_0 bounds a solid torus $\hat{N}_0 \subseteq N_1$. But $\hat{N}_0$ does not contain T_1 so $\hat{N}_0 = \hat{N}_1$. But then $f_*\colon \pi_1(\partial M) \longrightarrow \pi_1(\hat{N}_1)$ is not monic which is impossible. If T_0 is parallel to a component of ∂N_1, T_0 is parallel to a component of $\partial N_1 - (T_1 \cup T_2)$ so $\hat{N}_1$ is homeomorphic to $T \times [0, 1]$. This is impossible since $f\colon \partial M \longrightarrow N_1$ is not homotopic to a map into ∂N_1.

Thus we may assume that $f\colon \partial M \longrightarrow N_1$ is not homotopic to a map $f_1\colon \partial M \longrightarrow N_1$ such that $f_1(\partial M) \cap h(A)$ is empty.

After the usual argument, we may assume that $\partial M \cap f^{-1}h(A)$ is the union of a collection of disjoint, simple, essential loops $\lambda_1, \ldots, \lambda_m$. It can be seen that $f\colon \partial M \longrightarrow N_1$ is homotopic to a map $f_1\colon \partial M \longrightarrow N_1$ such that $f_1^{-1}h(A) = f_1^{-1}h(c_1) = f^{-1}h(A)$. Now the restriction of f_1 to the closures of the components of $\partial M - \bigcup_{i=1}^m \lambda_i$ determines a collection of maps $\hat{f}_i\colon (A, \partial A) \longrightarrow (N_1, T_1)$ for $i = 1, \ldots, m$ and it can be seen that if $\hat{f}_i$ is not essential for some i where

$1 \leq i \leq m$, $f_1: \partial M \longrightarrow N_1$ is not essential for it will be homotopic to a map into T_1. Thus by Theorem 2 there is an essential embedding $h_1: (A, \partial A) \longrightarrow (N_1, T_1)$.

It has been shown above that if $h(\partial A)$ is contained in a single component of $\partial N_1 - \partial N$, N_1 admits an essential embedding of T. This is a contradiction. Thus $f: \partial M \longrightarrow N_1$ is homotopic to a map $f_1: \partial M \longrightarrow \partial N_1$. This completes the proof of Theorem 9.

THEOREM 10. *Let M and N be compact connected, irreducible, boundary irreducible 3-manifolds. Suppose that ∂M is a torus and that M does not admit an essential embedding of an annulus. If $\Phi: \pi_1(M) \longrightarrow \pi_1(N)$ is an isomorphism, there is a homeomorphism h of M onto N such that $h_* = \Phi$.*

PROOF. By Theorem 9 there is a submanifold N_1 of N and covering map $\psi: M \longrightarrow N_1$ such that $\hat{\psi}_* = \Phi: \pi_1(M) \longrightarrow \pi_1(N)$. Thus $\hat{\psi}_*$ is an isomorphism so $\pi_1(N_1) \longrightarrow \pi_1(N)$ and $\psi_*: \pi_1(M) \longrightarrow \pi_1(N_1)$ are isomorphisms and ψ is an embedding. Let $N_2 = \mathrm{cl}(N - N_1)$. It is a consequence of van Kampen's theorem that $\pi_1(N)$ is the free product of $\pi_1(N_1)$ and $\pi_1(N_2)$ with amalgamation over $\pi_1(\partial N_1)$ since $\pi_1(\partial N_1) \longrightarrow \pi_1(N)$ is monic. It is a consequence of Lemma 4.2 in [2] that $\pi_1(\partial N_1) \longrightarrow \pi_1(N_2)$ is an epimorphism. It follows from Lemma 1.1 in [3] that N_2 is homeomorphic to $T \times [0, 1]$. Theorem 10 follows.

REMARK. Let M be the space of the square knot and N the space of the granny. Then it is known that $\pi_1(M) \cong \pi_1(N)$ but that M is not homeomorphic to N so the requirement in Theorem 10 that M does not admit an essential embedding of an annulus cannot be omitted.

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